Helium fine structure theory for the determination of α

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supported by NIST Precision measurement Grant

Introduction

- comparison of high precision theoretical predictions of atomic energy levels to experiments gives information on the structure of nuclei: rms charge radius, nuclear polarizability, magnetic moment, etc.
- determining fundamental constants from the atomic and molecular structure
- accurate treatment of electron correlations
- beyond static nucleus: finite nuclear mass corrections including relativistic effects

α from the fine structure in hydrogen

With the inclusion of the finite mass of the nucleus, relativistic, and QED corrections up to order $\alpha^3\nu_0$, the value of the fine structure constant is determined to be

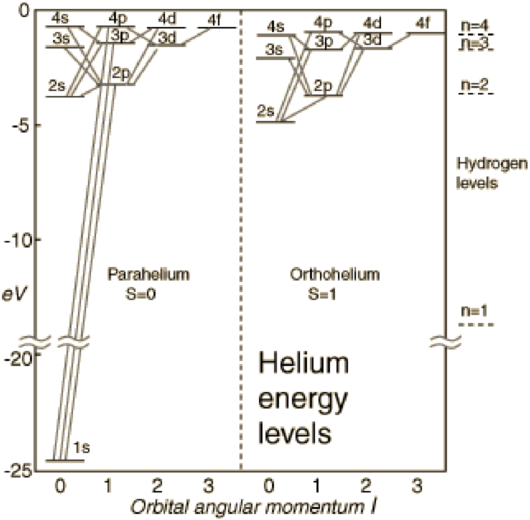
$$\alpha^{-1}(\mathrm{H}) = 137.035\,45(62).$$

It is consistent with the most accurately known value of α at present [Hanneke 2008, Kinoshita 2007]

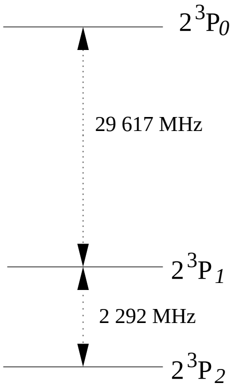
$$\alpha^{-1}(g{-}2) = 137.035\,999\,084(51).$$

The 4.5 ppm determination of α from hydrogen is much less precise than the 0.37 ppb value from *g*-2 because of the short lifetime of hydrogenic 2*p* states, of order 10^{-9} seconds.

Helium energy levels



Helium fine structure



- the large interval is used for determining α and the small one as a test of theory.

α from the helium fine structure

- determining α from the simplest many-electron atom, helium, program initiated in 1964 by Schwartz.
- the high experimental precision achieved in helium make possible accurate determination of α that depends on the low energy scales characteristic of atomic physics.
- three states is 2^3P_J, nonrelativistically degenerate, but relativistic effects lead to a frequency splitting of order $\alpha^2 R_\infty c$.

Expansion of binding energy in α

Expansion of the energy in powers of the fine structure constant ($\alpha \approx 1/137$)

$$E_{\rm fs}(\alpha) = E_{\rm fs}^{(4)} + E_{\rm fs}^{(6)} + E_{\rm fs}^{(7)} + \cdots$$

- $E^{(n)} \sim \alpha^n \mathcal{E}^{(n)}$
- Valid for small systems with not too large nuclear charge Z
- Expansion coefficients are expressed in terms of nonrelativistic expectation values of effective Hamiltonians
- $E_{\rm fs}^{(4)} = \langle H_{\rm fs} \rangle$

Leading relativistic, QED and the nuclear recoil

$$
\begin{aligned}
H_{\rm fs} \;=\;& \frac{\alpha}{4\,m^2}\left(\frac{\vec\sigma_1\cdot\vec\sigma_2}{r^3} - 3\,\frac{\vec\sigma_1\cdot\vec r\,\vec\sigma_2\cdot\vec r}{r^5}\right)(1+a_e)^2 \\
&+ \frac{Z\alpha}{4m^2}\left[\frac{1}{r_1^3}\,\vec r_1\times\vec p_1\cdot\vec\sigma_1 + \frac{1}{r_2^3}\,\vec r_2\times\vec p_2\cdot\vec\sigma_2\right](1+2a_e) \\
&+ \frac{\alpha}{4\,m^2\,r^3}\Big[\big[(1+2\,a_e)\,\vec\sigma_2 + 2\,(1+a_e)\,\vec\sigma_1\big]\cdot\vec r\times\vec p_2 \\
&\quad -\big[(1+2\,a_e)\,\vec\sigma_1 + 2\,(1+a_e)\,\vec\sigma_2\big]\cdot\vec r\times\vec p_1\Big] \\
&+ \frac{Z\alpha}{2mM}\left[\frac{\vec r_1}{r_1^3}\times(\vec p_1+\vec p_2)\cdot\vec\sigma_1 + \frac{\vec r_2}{r_2^3}\times(\vec p_1+\vec p_2)\cdot\vec\sigma_2\right](1+a_e)
\end{aligned}
$$

Higher order corrections

$$
\begin{aligned}
E^{(6)} &= \langle H^{(6)}\rangle + \langle H^{(4)}\,\frac{1}{(E_0-H_0)'}\,H^{(4)}\rangle \\
E^{(7)} &= \langle H^{(7)}\rangle + 2\,\left\langle H^{(5)}\,\frac{1}{(E_0-H_0)'}\,H^{(4)}_{\rm fs}\right\rangle + E_L \\
H^{(5)} &= -\frac{7}{6\,\pi}\,\frac{\alpha^2}{r^3} + \frac{38\,Z\,\alpha^2}{45}\,\left[\delta^3(r_1)+\delta^3(r_2)\right]
\end{aligned}
$$

- anomalous magnetic moment
- electron self-energy and vacuum-polarization
- finite nuclear mass effects

Higher order effective Hamiltonian $H^{(7)}$

$$
\begin{aligned}
H^{(7)} &= Z\,\alpha^7\left(\frac{91}{180}+\frac{2}{3}\ln[(Z\,\alpha)^{-2}]\right) i\,\vec{p}_1\times\delta^3(r_1)\,\vec{p}_1\cdot\vec{\sigma}_1 \\
&+\alpha^7\left(-\frac{83}{60}+\frac{\ln\alpha}{2}\right)(\vec{\sigma}_1\cdot\vec{\nabla})\,(\vec{\sigma}_2\cdot\vec{\nabla})\delta^3(r) \\
&-\alpha^7\,\frac{15}{8\,\pi}\,\frac{1}{r^7}\,(\vec{\sigma}_1\cdot\vec{r})\,(\vec{\sigma}_2\cdot\vec{r}) \\
&+\alpha^7\left(\frac{69}{10}+3\,\ln\alpha\right) i\,\vec{p}_1\times\delta^3(r)\,\vec{p}_1\cdot\vec{\sigma}_1 \\
&-\alpha^7\,\frac{3}{4\,\pi}\,i\,\vec{p}_1\times\frac{1}{r^3}\,\vec{p}_1\cdot\vec{\sigma}_1+H_{\rm amm}
\end{aligned}
$$

- dimensional regularization

Nonrelativistic wave function

- $\vec{\phi}(\vec{r}_1, \vec{r}_2) = \sum_{i=1}^{N} c_i \left[\vec{r}_1 \exp(-\alpha_i\, r_1 - \beta_i\, r_2 - \gamma_i\, r) - (1 \leftrightarrow 2)\right]$
- variational approach: minimize energy with respect to $c_i, \alpha_i, \beta_i, \gamma_i$
- master integral

$$\frac{1}{16\,\pi^2} \int d^3 r_1 \int d^3 r_2\, \frac{e^{-\alpha r_1 - \beta r_2 - \gamma r_{12}}}{r_1\, r_2\, r_{12}} = \frac{1}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}$$

- parameters α_i, β_i, and γ_i are chosen quasirandomly

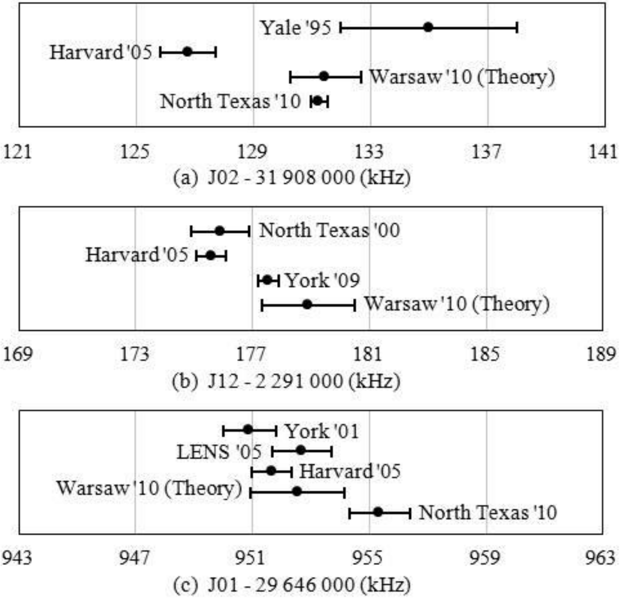
$$\begin{aligned} \alpha_i &\in [A_1, A_2] \\ \beta_i &\in [B_1, B_2] \\ \gamma_i &\in [C_1, C_2] \end{aligned}$$

- $E_0(2^3P) = -2.133\,164\,190\,779\,283\,205\,146\,96\,^{+0}_{-10}$

Results

Term	ν_{01}	ν_{12}	Ref.
$m\alpha^4(+m/M)$	29 563 765.45	2 320 241.43	
	29 563 765.23	2 320 241.42	Drake (2002)
$m\alpha^5(+m/M)$	54 704.04	$-22\,544.00$	
	54 704.04	$-22\,545.01$	Drake (2002)
$m\alpha^6$	$-1\,607.52(2)$	$-6\,506.43$	
	$-1\,607.61(4)$	$-6\,506.45(7)$	Drake (2002)
$m\alpha^6 m/M$	-9.96	9.15	
	$-10.37(5)$	9.80(11)	Drake (2002)
$m\alpha^7\log(Z\alpha)$	81.43	-5.87	
	81.42	-5.87	Drake (2002)
$m\alpha^7$, nlog	18.86	-14.38	
$m\alpha^8$	± 1.7	± 1.7	
Total theory	$29\,616\,952.29 \pm 1.7$	$2\,291\,178.91 \pm 1.7$	

Comparison with experiments: from Shiner 2010



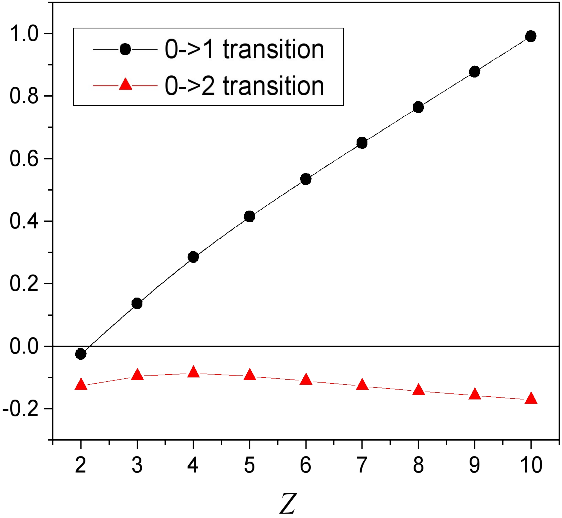
Determination of the fine structure constant

- α from ν_{01} Zelevinsky, Farkas, and Gabrielse (2005)

  $\alpha^{-1}(\mathrm{He}) = \quad 137.036\,001\,1(39)_{\mathrm{theo}}(16)_{\mathrm{exp}}$

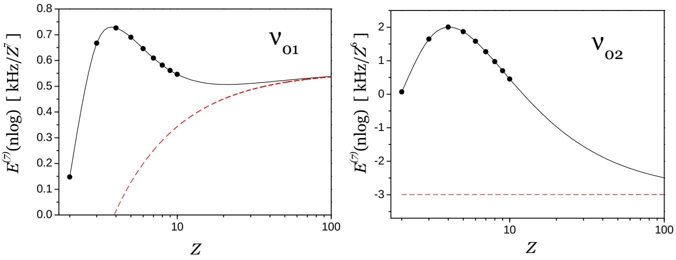
- $\alpha^{-1}(g-2) = 137.035\,999\,084(51)$
- theoretical uncertainty is due to the higher order terms
- ν_{02} fine structure measurement in heliumlike ions ?

$m\alpha^6$ correction im MHz/Z^6



Tests

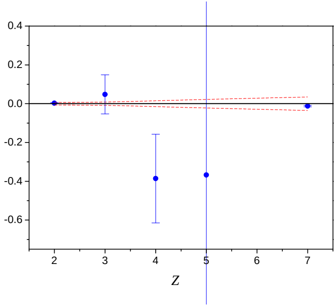
- checking the hydrogenic limit



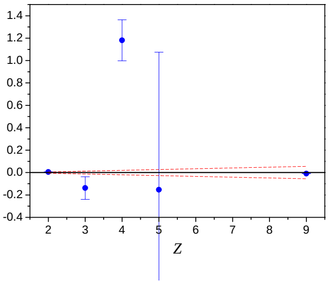
- comparison with experiment for different nuclear charges

Differences: theory-experiment in kHz/Z^8

$0 \rightarrow 1$ transition



$1 \rightarrow 2$ transition



$Z = 3$ Riis *et al.* 1994

$Z = 4$ Scholl *et al.* 1993

$Z = 5$ Dinneen *et al.* 1991

$Z = 7$ Thompson *et al.* 1998

$Z = 9$ Myers *et al.* 1999

Conclusions

- $\alpha^{-1}(\mathrm{He}) =$ 137.035 999 5(39)$_{\mathrm{theo}}$(6)$_{\mathrm{exp}}$ from Shiner 2010
- disagreement between experimental results for ν_{12} and ν_{02}
- large uncertainty due to higher order terms
- testing $1/Z$ expansion against hydrogenic limit
- possible 10^{-9} determination of α requires more accurate estimation of higher order terms
- one of the most accurate tests of QED, (μH+H Lamb shift)