Faculty of Information and Communication Technologies

On Detecting Double Literal Faults

Technical Report: SUTICT-TR2006.04

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First version: 30th June 2007

Revised: 30th Nov 2007 [Renumber the expression ids]

Last revised: 15th Oct 2008 [Fixing minor errors, New results - Theorems 5.7 and 5.9 in Section 5.3]

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1 Introduction

The aim of software testing is to reveal errors in a program. Normally, after performing testing on a program, we still cannot say the program is fault free. However, fault-based testing is proposed as a technique to detect hypothesized fault in the program under test. More precisely, when test cases generated by fault-based testing strategies cannot reveal program failures due to the hypothesized fault, we can claim that the hypothesized fault does not exist in the program. Many fault-based testing techniques have been proposed [2, 3, 11, 12, 17].

Recently, the detection conditions of hypothesized faults have been studied [1, 5, 9, 15]. The detection conditions of hypothesized faults are mainly used in two perspectives. First, they are used to develop test case selection strategies to detect particular types of faults. In [1], Chen and Lau proposed three test case selection strategies based on fault detection conditions of seven types of faults. Second, the detection conditions are used to build *fault class hierarchy*. Fault class hierarchy establishes relationships between different types of fault classes. For example, if any test case that can detect fault class A can also detect fault class B, then A is put in the lower part of the hierarchy tree than that of B. Kuhn [5] and Lau and Yu [9] used fault class hierarchies to explain the empirical results for existing fault-based testing methodologies.

Most research on detection conditions assume that various types of hypothesized faults may occur in a program. However, when they study the fault detection conditions, they assume that only a single hypothesized fault can occur in the program.

On the contrary, many empirical studies on software faults show that, in practice, multiple faults occur more often in programs [10, 16]. Previous research on multiple faults [4, 12] studied the fault coupling effect of double faults. Double faults being occurred in a program is a special instance of multiple faults. When two faults combined in such a way that they cannot be detected using test cases that detect the faults in isolation, we say that the two faults are *coupled* together. The study of fault coupling effect of double faults is mainly from the empirical perspective and is on the chances of a test set can detect the double faults provided that it detects two individual faults in isolation.

Recently, Lau and Yu extended their study on using fault class hierarchy to study double faults related to terms [8]. They found that test case that detects some particular classes of faults in the lower part of the hierarchy can detect double faults which involve these classes of faults and faults in the upper part of the hierarchy. For example, if a test case that can detect fault class A which is lower than fault class B in the fault class hierarchy, then the same test case can detect the double fault involving fault classes A and B, denoted as $A \bowtie B$. In [7], Lau *et al.* further study the detection conditions of double faults related to terms. Five different types of single term faults are considered. They found that all double faults formed by these single term faults can be guaranteed to be detected by any test case selection strategy that subsumes the BASIC meaningful impact strategy proposed in [17].

In this report, we study the detection conditions of double faults related to literals. Different from the results in double faults related to terms, our analysis of the detection condition of double fault related to literals shows that existing test case selection strategies are insufficient in detecting double faults related to literal. Therefore, new test case selection strategies have been proposed to supplement existing test case selection strategies to detect the studied double faults.

The rest of the report is organized as follows. Section 2 introduces the notation and fault classes studied in this report. Section 3 presents double fault classes and their corresponding faulty implementations. Section 4 analyses fault detection conditions of studied double faults classes. Section 5 investigates the existing test case selection strategies in detecting double faults classes related to literals. Section 6 proposes a family of test case selection strategies. Section 7 concludes the report.

2 Preliminary

2.1 Notation

In this report, we use '·', '+' and '¯' to represent Boolean operators, AND, OR and NOT, respectively. Usually, '·' is omitted whenever it is clear from the context. We use 1 and 0 to represent the truth values 'TRUE' and 'FALSE', respectively. The set of all truth values, that is $\{0,1\}$, is denoted by $\mathbb{B}$.

Let S be a Boolean expression in disjunctive normal form

$$S = p_1 + \cdots + p_m$$

where m is the number of terms, $p_i = x_1^i \cdots x_{k_i}^i$ is the i-th term of S, x_j^i is the j-th literal in p_i, and k_i is the number of literals in p_i. A Boolean expression is in *irredundant disjunctive normal form* if (1) none of its terms can be omitted from the expression; and (2) none of its literals can be omitted from any term in the expression.

Let S be a Boolean expression having n variables, the input domain is the n-dimensional Boolean space $\mathbb{B}^n$. True points are those that cause S evaluates to 1. The set of all true points of S is denoted by $TP(S)$. A true point of the term p_i in S is a point that makes p_i evaluates to 1. The set of all true points of p_i in S is denoted by $TP_i(S)$. Hence, $TP(S) = \bigcup_i TP_i(S)$. A *unique true point* of p_i in S is a true point of S that makes (1) p_i evaluate to 1; and (2) all other terms evaluate to 0. The set of all unique true points of p_i in S is denoted by $UTP_i(S)$. The set of all unique true points of S is denoted by $UTP(S)$ and $UTP(S) = \bigcup_i UTP_i(S)$.

False points of S are those that make S evaluates to 0 and the set of all false points is denoted by $FP(S)$. A *near false point* of the j-th literal x_j^i of the i-th term p_i in S is a false point that makes (1) x_j^i evaluates to 0, and (2) all other literals in p_i evaluate to 1. The set of all near false points for the j-th literal x_j^i of the i-th term p_i in S is denoted by $NFP_{i,\bar{j}}(S)$. The set of all near false points for the i-th term p_i in S is denoted by $NFP_i(S)$. Therefore, $NFP_i(S)$= $\bigcup_j NFP_{i,\bar{j}}(S)$. The set of all near false points of S is denoted by $NFP(S)$ and $NFP(S) = \bigcup_i NFP_i(S)$.

2.2 Fault Classes

In this report, we only consider four fault classes related to literals in a Boolean expression. Let S be a Boolean expression S in irredundant disjunctive normal form. Suppose a fault F changes a subexpression E into a subexpression E'. The resulting faulty implementation, referred as *single-fault expression*, is denoted by $I_{F(E \to E')}$. The single-fault expression differs from the original expression by one syntactic change and is not equivalent to the original expression. The following four fault classes related to literals in a Boolean expression are studied in this report:

1. *Literal Negation Fault* (LNF): A literal in a particular term in the Boolean expression is replaced by its negation. For example, the Boolean expression $ab + cd + ef$ may be wrongly implemented as $\bar{a}b + cd + ef$. If the literal x^i_j of the i-th term, p_i of S is wrongly implemented as its negation where $1 \leq j \leq k_i$ and k_i is the number of literals in p_i. The implementation, denoted by $I_{LNF(p_i \to p_{i,\bar{j}})} = p_1 + \cdots + p_{i-1} + p_{i,\bar{j}} + p_{i+1} + \cdots + p_m$. As reported in [9], when p_i contains just one literal, the negation fault is considered as a term negation fault rather than a literal negation fault.

2. *Literal Omission Fault* (LOF): A literal in a particular term in the Boolean expression is omitted. For example, the Boolean expression $ab + cd + ef$ may be implemented as $ab + cd + e$. If the literal x^i_j of the i-th term, p_i of S is omitted where $1 \leq j \leq k_i$ and k_i is the number of literals in p_i, the implementation is then equivalent to $I_{LOF(p_i \to p_{i,\hat{j}})} = p_1 + \cdots + p_{i-1} + p_{i,\hat{j}} + p_{i+1} + \cdots + p_m$. As reported in [9], when p_i contains just one literal, the omission fault is considered as a term omission fault rather than a literal omission fault.

3. *Literal Insertion Fault* (LIF): A literal not appearing in a particular term of a Boolean expression is inserted into that term. For example, the Boolean expression $ab + cd + ef$ may be

implemented as $abc+cd+ef$. If the literal x_l which does not appear in the i-th term p_i of S (that is, $x_l, \bar{x}_l \notin x_1^i, \ldots x_{k_i}^i$) and is inserted into p_i where k_i is the number of literals of p_i, the implementation is then equivalent to $I_{LIF(p_i \rightarrow p_i x_l)} = p_1 + \cdots + p_{i-1} + p_i x_l + p_{i+1} + \cdots + p_m$.

4. *Literal Reference Fault* (LRF): A literal in a particular term of a Boolean specification is replaced by another literal not appearing in the term during the implementation. For example, the Boolean expression $ab+cd+ef$ may be implemented as $ac+cd+ef$. If the literal x_j^i in the i-th term p_i of S is replaced by the literal x_l which does not appear in p_i ($x_l, \bar{x}_l \notin x_1^i, \ldots x_{k_i}^i$) where $1 \leq j \leq k_i$ and k_i is the number of literals in p_i, the implementation is then equivalent to $I_{LRF(p_i \rightarrow p_{i,\hat{j}} x_l)} = p_1 + \cdots + p_{i-1} + p_{i,\hat{j}} x_l + p_{i+1} + \cdots + p_m$.

2.3 Existing Strategy for Single Fault Detection

There are many test case selection strategies for detecting single faults in Boolean expressions including the BOR strategy [14, 13], the BASIC, MAX-A and MAX-B meaningful impact strategies (or simply the BASIC, the MAX-A and the MAX-B strategies) [17] and the MUMCUT strategy [18]. Since the BOR strategy requires every variable in the expression to occur only once, it is not widely applicable to all Boolean expressions in IDNF.

In the following, we review the four strategies studied in this report, namely, the BASIC, MUMCUT, MAX-A and MAX-B strategies. Let $S(= p_1 + \cdots + p_m)$ be a Boolean expression in IDNF.

1. The BASIC strategy requires to select (1) one point from $UTP_i(S)$ for every i; and (2) one point from $NFP_{i,\bar{j}}(S)$ for every possible i and j pair.

2. The MUMCUT strategy is a combination of three test case selection strategies which are MUTP, MNFP and CUTPNFP strategies. The three strategies are:

   (a) MUTP strategy requires to select unique true points from $UTP_i(S)$ such that all possible truth values (that is, 0 and 1) of every literal not occurring in p_i are covered, for every i.

(b) MNFP strategy requires to select near false points from $NFP_{i,\bar{j}}(S)$ such that all possible truth values of every literal not occurring in p_i are covered, for every possible i and j pair.

(c) CUTPNFP strategy requires to select one unique true point from $UTP_i(S)$ and one near false point from $NFP_{i,\bar{j}}(S)$ such that they only differ at the truth value of j-th literal of p_i, for every possible i and j pair.

3. The MAX-A strategy requires to select (1) all points from $UTP_i(S)$ for every i; and (2) all points from $NFP_{i,\bar{j}}(S)$ for every possible i and j pair.

4. The MAX-B strategy requires to select (1) all points from $UTP_i(S)$ for every i; (2) all points from $NFP_{i,\bar{j}}(S)$ for every possible i and j pair; (3) $\lceil \log_2(|OTP(S)|) \rceil$ points from $OTP(S)$ where $|OTP(S)|$ denotes the size of $OTP(S)$ (one point is selected if $OTP(S)$ is a singleton set); and (4) $\lceil \log_2(|RFP(S)|) \rceil$ points from $RFP(S)$ where $|RFP(S)|$ denotes the size of $RFP(S)$ (one point is selected if $RFP(S)$ is a singleton set).

It should be noted that the MAX-B strategy subsumes the MAX-A strategy, which in turn subsumes the MUMCUT strategy; which in turn subsumes the BASIC strategy. A test selection criterion A is said to *subsume* another test selection criterion B if a test set satisfying A always satisfies B.

It has been shown in [1] that the BASIC strategy cannot guarantee to detect LIF and LRF. However, in [7], the BASIC strategy, and hence any strategy that subsumes it, can guarantee to detect all double faults related to terms.

3 Double Faults

Multiple occurrences of fault classes may result in faulty expressions which differ from the original expression with several syntactic changes. For example, $abce + ade$ differs from $abc + de$ by two syntactic changes. An expression which differs from the original expression by more than one syntactic change is said to contain multiple faults. In this paper, *double literal faults* are defined as two occurrences of single literal faults.

There are possible two situations when two single faults are committed in a program. For the first situation, the two faults, no matter which occur first, will result in the same faulty expression. Such situation is referred as *double fault without ordering*. The second situation, the first fault may affect the occurrence of the second fault, therefore, the resulting expression may differ once the order of the two faults is changed.

As reported in [6], of the four single fault classes studied in this report, there are 10 classes of double faults without ordering resulting in 19 different *double-fault expressions*. A double-fault expression is an expression that (1) differs from the original expression by two syntactic changes and (2) is equivalent to neither the original expression nor any faulty expression with a single fault. For the case of double faults with ordering, there are 16 double fault classes resulting in 30 possible double-fault expressions. Lau and Liu [6] found that two faulty expressions of double fault without ordering are actually equivalent. Moreover, all 30 possible faulty expressions have their corresponding counterparts in double fault without ordering. Hence, the 19 different double-fault expressions due to double fault without ordering can represent all double faults related to literal [6].

For any two single fault classes A and B, we use the notation $A \bowtie B$ to denote the double fault class formed from A and B, that is, the class of faults due to the occurrences of two faults: one fault of class A and another fault of class B. Given a Boolean expression S, suppose two faults A and B are committed on the expression changing E_1 and E_2 in S to E_1' and E_2', respectively, the resulting faulty expression (or, implementation) is denoted as $I_{F_1(E_1 \to E_1') \bowtie F_1(E_2 \to E_2')}$.

Table 1 shows the double fault classes and their corresponding faulty expressions. Let us consider the row corresponding to LNF $\bowtie$ LOF in Table 1. Let S be a Boolean expression in IDNF. There are two subcases. First, the literal $x_{j_1}^{i_1}$ of the i_1-th term, p_{i_1}, in S is negated and the literal $x_{j_2}^{i_2}$ of the i_2-th term, p_{i_2}, in S is omitted. Without loss of generality, we may assume that $i_1 < i_2$. The double-fault expression is equivalent to $p_1 + \cdots + p_{i_1,\bar{j}_1} + \cdots + p_{i_2,\hat{j}_2} + \cdots + p_m$. Second, the literal $x_{j_1}^{i_1}$ of the i_1-th term, p_{i_1}, in S is negated and the literal $x_{j_2}^{i_1}$ of p_{i_1} omitted. Without loss of generality, we may assume that $j_1 \leq j_2$. The faulty expression is then equivalent to $p_1 + \cdots + p_{i_1,\bar{j}_1,\hat{j}_2} + \cdots + p_m$ where $p_{i_1,\bar{j}_1,\hat{j}_2}$ denotes the term obtained from p_{i_1} by negating its j_1-th literal and omitting its j_2-th literal.

4 Detection Conditions of Double Faults

The detection conditions of hypothesized faults have been studied recently [1, 5, 9, 15]. Let S be a specification and I be the expression which differs from S by several syntactic changes. Whenever S and I evaluate to different values, they can be distinguished from each other. The *detection condition* of I with respect to S is a condition that makes S and I evaluate to different values. As a result, the Boolean exclusive-or operator XOR, denoted as $\oplus$, can be used to find the detection conditions. In short, the detection condition can be derived from $S \oplus I$.

In this report, we will concentrate on double faults related to literals. As discussed previously, 19 double-fault expressions described in Table 1 can represent all double faults related to literals. In the rest of this section, we consider the detection conditions of these double-fault expressions. Instead of simply presenting the Boolean expression $S \oplus I$ as detection conditions, we present them as conditions satisfied by test cases in $\mathbb{B}^n$. Since such categorization is based on certain properties of test sets, it helps to identify and develop test case selection strategy to detect such double faults in Table 1.

Table 1: Double fault and double-fault expression ($S = p_1 + \ldots + p_m$)

Fault Class	Double-fault Expression	
LNF ⋈ LNF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1,\bar{j}_1} + \cdots + p_{i_2,\bar{j}_2} + \cdots + p_m$	(1)
	Case 2 ($j_1 < j_2$): $p_1 + \cdots + p_{i_1,\bar{j}_1,\bar{j}_2} + \cdots + p_m$	(2)
LNF ⋈ LOF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1,\bar{j}_1} + \cdots + p_{i_2,\hat{j}_2} + \cdots + p_m$	(3)
	Case 2 ($j_1 < j_2$): $p_1 + \cdots + p_{i_1,\bar{j}_1,\hat{j}_2} + \cdots + p_m$	(4)
LNF ⋈ LIF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1,\bar{j}_1} + \cdots + p_{i_2}x_{l_2} + \cdots + p_m$	(5)
	Case 2: $p_1 + \cdots + p_{i_1,\bar{j}_1}x_{l_2} + \cdots + p_m$	(6)
LNF ⋈ LRF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1,\bar{j}_1} + \cdots + p_{i_2,\hat{j}_2}x_{l_2} + \cdots + p_m$	(7)
	Case 2 ($j_1 < j_2$): $p_1 + \cdots + p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2} + \cdots + p_m$	(8)
LOF ⋈ LOF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1,\hat{j}_1} + \cdots + p_{i_2,\hat{j}_2} + \cdots + p_m$	(9)
	Case 2 ($j_1 < j_2$): $p_1 + \cdots + p_{i_1,\hat{j}_1,\hat{j}_2} + \cdots + p_m$	(10)
LOF ⋈ LIF	($i_1 < i_2$): $p_1 + \cdots + p_{i_1,\bar{j}_1} + \cdots + p_{i_2}x_{l_2} + \cdots + p_m$	(11)
LOF ⋈ LRF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1,\hat{j}_1} + \cdots + p_{i_2,\hat{j}_2}x_{l_2} + \cdots + p_m$	(12)
	Case 2 ($j_1 < j_2$): $p_1 + \cdots + p_{i_1,\hat{j}_1,\hat{j}_2}x_{l_2} + \cdots + p_m$	(13)
LIF ⋈ LIF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1}x_{l_1} + \cdots + p_{i_2}x_{l_2} + \cdots + p_m$	(14)
	Case 2: $p_1 + \cdots + p_{i_1}x_{l_1}x_{l_2} + \cdots + p_m$	(15)
LIF ⋈ LRF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1}x_{l_1} + \cdots + p_{i_2,\hat{j}_2}x_{l_2} + \cdots + p_m$	(16)
	Case 2: $p_1 + \cdots + p_{i_1,\hat{j}_2}x_{l_1}x_{l_2} + \cdots + p_m$	(17)
LRF ⋈ LRF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1,\hat{j}_1}x_{l_1} + \cdots + p_{i_2,\hat{j}_2}x_{l_2} + \cdots + p_m$	(18)
	Case 2 ($j_1 < j_2$): $p_1 + \cdots + p_{i_1,\hat{j}_1,\hat{j}_2}x_{l_1}x_{l_2} + \cdots + p_m$	(19)

4.1 LNF with other Faults

Theorem 4.1 *(LNF with LNF - Case 1)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two literals $x^{i_1}_{j_1}$ in the i_1-th term, p_{i_1}, in S and $x^{i_2}_{j_2}$ in the i_2-th term, p_{i_2}, in S are negated where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_1 \leq k_{i_1}$, $1 \leq j_2 \leq k_{i_2}$ and $k_{i_1}(>1)$ and $k_{i_2}(>1)$ are the numbers of literals of p_{i_1} and p_{i_2}, respectively, the resulting expression denoted as $I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \rtimes LNF(p_{i_2} \to p_{i_2,\bar{j}_2})}$ is equivalent to that given by double-fault expression (1) in Table 1. Then, $S \not\equiv I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \rtimes LNF(p_{i_2} \to p_{i_2,\bar{j}_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ *such that* $p_{i_2,\bar{j}_2}=0$,
2. $\vec{t} \in TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ *such that* $p_{i_1,\bar{j}_1}=0$,
3. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, *or*
4. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$.

Proof : First, we observe that $S \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \rtimes LNF(p_{i_2} \to p_{i_2,\bar{j}_2})}$

$$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1,\bar{j}_1} + p_{i_2,\bar{j}_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1,\bar{j}_1} + p_{i_2,\bar{j}_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1,\bar{j}_1} + p_{i_2,\bar{j}_2})\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2})\bar{p}_{i_1,\bar{j}_1}\bar{p}_{i_2,\bar{j}_2} + \bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1,\bar{j}_1} + p_{i_2,\bar{j}_2})\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv (p_{i_1}\bar{p}_{i_2,\bar{j}_2} + \bar{p}_{i_1,\bar{j}_1}p_{i_2} + p_{i_1,\bar{j}_1}\bar{p}_{i_1}\bar{p}_{i_2} + p_{i_2,\bar{j}_2}\bar{p}_{i_1}\bar{p}_{i_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

(By making use of $(AB)(\overline{\bar{A}\bar{B}}) \equiv AB$)

$$\equiv p_{i_1}\bar{p}_{i_2,\bar{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{p}_{i_1,\bar{j}_1}p_{i_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$+ p_{i_1,\bar{j}_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_2,\bar{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv p_{i_1}\bar{p}_{i_2,\bar{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{p}_{i_1,\bar{j}_1}p_{i_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$+ p_{i_1,\bar{j}_1}\overline{S} + p_{i_2,\bar{j}_2}\overline{S}$$

Now, $S(\vec{t}) \neq I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LNF(p_{i_2} \to p_{i_2,\bar{j}_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LNF(p_{i_2} \to p_{i_2,\bar{j}_2})}(\vec{t}) = 1$

if and only if $p_{i_1}\bar{p}_{i_2,\bar{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{p}_{i_1,\bar{j}_1} p_{i_2}\bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S} + p_{i_2,\bar{j}_2}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S)\big)$ such that $p_{i_2,\bar{j}_2}$=0,
2. $\vec{t} \in TP_{i_2}(S) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S)\big)$ such that $p_{i_1,\bar{j}_1}$=0,
3. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or
4. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$.

Hence, the result follows. □

Theorem 4.2 *(LNF with LNF - Case 2)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two literals $x^{i_1}_{j_1}$ and $x^{i_1}_{j_2}$ in the i_1-th term, p_{i_1}, in S are negated where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 \leq k_{i_1}$ and $k_{i_1} (> 1)$ is the number of literals of p_{i_1}, the resulting expression denoted as $I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LNF(p_{i_1} \to p_{i_1,\bar{j}_2})}$ is equivalent to double-fault expression (2) in Table 1. Then, $S \not\equiv I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LNF(p_{i_1} \to p_{i_1,\bar{j}_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S)$, or*
2. *$\vec{t} \in FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2}$ = 1.*

where $p_{i_1,\bar{j}_1,\bar{j}_2}$ denotes the term obtained from p_{i_1} by negating its j_1-th and j_2-th literals.

Proof : First, we observe that $S \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LNF(p_{i_1} \to p_{i_1,\bar{j}_2})}$

$\equiv (p_{i_1} \oplus p_{i_1,\bar{j}_1,\bar{j}_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1}\bar{p}_{i_1,\bar{j}_1,\bar{j}_2} + \bar{p}_{i_1}p_{i_1,\bar{j}_1,\bar{j}_2}\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv (p_{i_1} + \bar{p}_{i_1}p_{i_1,\bar{j}_1,\bar{j}_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

(By making use of $(ABC)(\overline{A \cdot \overline{B} \cdot \overline{C}}) \equiv ABC$)

$\equiv p_{i_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1,\bar{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1,\bar{j}_2}\overline{S}$

Now,	$S(\vec{t}) \neq I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LNF(p_{i_1} \to p_{i_1,\bar{j}_2})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LNF(p_{i_1} \to p_{i_1,\bar{j}_2})}(\vec{t}) = 1$
if and only if	$p_{i_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1,\bar{j}_2}\overline{S}$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$, or
2. $\vec{t} \in FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} = 1$.

Hence, the result follows. □

Theorem 4.3 *(LNF with LOF - Case 1)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal $x^{i_1}_{j_1}$ in the i_1-th term, p_{i_1}, in S is negated and the literal $x^{i_2}_{j_2}$ in the i_2-th term, p_{i_2}, in S is omitted from p_{i_2} where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_1 \leq k_{i_1}$, $1 \leq j_2 \leq k_{i_2}$, and $k_{i_1}(> 1)$ and $k_{i_2}(> 1)$ are the numbers of literals of p_{i_1} and p_{i_2}, respectively, the resulting expression denoted as $I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_2} \to p_{i_2,\hat{j}_2})}$ is equivalent to double-fault expression (3) in Table 1. Then, $S \not\equiv I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_2} \to p_{i_2,\hat{j}_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} = 0$,*

2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$*, or*

3. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$*.*

Proof : First, we observe that $S \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_2} \to p_{i_2,\hat{j}_2})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1,\bar{j}_1} + p_{i_2,\hat{j}_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1,\bar{j}_1} + p_{i_2,\hat{j}_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1,\bar{j}_1} + p_{i_2,\hat{j}_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\bar{p}_{i_1,\bar{j}_1}\bar{p}_{i_2,\hat{j}_2}) + \bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1,\bar{j}_1} + p_{i_2,\hat{j}_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv (p_{i_1}\bar{p}_{i_2,\hat{j}_2} + 0 + p_{i_1,\bar{j}_1}\bar{p}_{i_1}\bar{p}_{i_2} + p_{i_2,\bar{j}_2}\bar{p}_{i_1}\bar{p}_{i_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $(AB)(\overline{A\bar{B}}) \equiv AB$ and $AB(\overline{A}) \equiv 0$)

$\equiv (p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{p}_{i_2} + p_{i_1,\bar{j}_1}\bar{p}_{i_1}\bar{p}_{i_2} + p_{i_2,\bar{j}_2}\bar{p}_{i_1}\bar{p}_{i_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

(By rewriting $\overline{A}$ as $(\overline{A})(\overline{AB})$ because they are equivalent)

$\equiv p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+ p_{i_2,\bar{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S} + p_{i_2,\bar{j}_2}\overline{S}$

Now, $S(\vec{t}) \neq I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_2} \to p_{i_2,\hat{j}_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_2} \to p_{i_2,\hat{j}_2})}(\vec{t}) = 1$

if and only if $p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S} + p_{i_2,\bar{j}_2}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} = 0$,
2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or
3. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$.

Hence, the result follows. □

Theorem 4.4 *(LNF with LOF - Case 2)*

Let S*=*$p_1 + \cdots + p_m$ *be a Boolean specification in irredundant disjunctive normal form. Suppose* *that the literal $x^{i_1}_{j_1}$ in the i_1-th term, p_{i_1}, in S is negated and the literal $x^{i_1}_{j_2}$ in the i_1-th term, p_{i_1}, in S is omitted where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 \leq k_{i_1}$ and $k_{i_1}(> 1)$ is the number of literals of p_{i_1}, the resulting expression denoted as $I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_1} \to p_{i_1,\hat{j}_2})}$ is equivalent to that given by double-fault expression (4) in Table 1. Then, we have $S \not\equiv I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_1} \to p_{i_1,\hat{j}_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:*

1. *$\vec{t} \in UTP_{i_1}(S)$, or*
2. *$\vec{t} \in FP(S)$ such that $p_{i_1,\bar{j}_1,\hat{j}_2} = 1$.*

where $p_{i_1,\bar{j}_1,\hat{j}_2}$ denotes the term obtained from p_{i_1} by negating its j_1-th literal and omitting its j_2-th literal.

Proof : First, we observe that $S \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_1} \to p_{i_1,\hat{j}_2})}$

$\equiv (p_{i_1} \oplus p_{i_1,\bar{j}_1,\hat{j}_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1}\bar{p}_{i_1,\bar{j}_1,\hat{j}_2} + \bar{p}_{i_1}p_{i_1,\bar{j}_1,\hat{j}_2}\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv (p_{i_1} + p_{i_1,\bar{j}_1,\hat{j}_2}\bar{p}_{i_1})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

(By making use of $(ABC)\overline{A\overline{B}} \equiv ABC$)

$\equiv p_{i_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1,\hat{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1,\hat{j}_2}\overline{S}$

Now, $S(\vec{t}) \neq I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_1} \to p_{i_1,\hat{j}_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LOF(p_{i_1} \to p_{i_1,\hat{j}_2})}(\vec{t}) = 1$

if and only if $p_{i_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1,\hat{j}_2}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$, or
2. $\vec{t} \in FP(S)$ such that $p_{i_1,\bar{j}_1,\hat{j}_2} = 1$.

Hence, the result follows. □

Theorem 4.5 *(LNF with LIF - Case 1)*

Let S*=*$p_1 + \cdots + p_m$ *be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal* $x^{i_1}_{j_1}$ *in the* i_1*-th term,* p_{i_1}*, in* S *is negated and the literal* x_{l_2} *is inserted in the* i_2*-th term,* p_{i_2}*, in* S *where* $1 \le i_1 < i_2 \le m$*,* $1 \le j_1 \le k_{i_1}$*,* $k_{i_1}(>1)$ *is the number of literals of* p_{i_1} *and* x_{l_2} *is a missing literal of* p_{i_2}*, the resulting expression denoted as* $I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_2} \to p_{i_2}x_{l_2})}$ *is equivalent to that given by double-fault expression (5) in Table 1. Then,* $S \not\equiv I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_2} \to p_{i_2}x_{l_2})}$ *if and only if there is a test case* $\vec{t}$ *that satisfies any of the following conditions:*

1. $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ *such that* $p_{i_2}x_{l_2} = 0$*,*
2. $\vec{t} \in TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ *such that* $p_{i_1,\bar{j}_1} + x_{l_2} = 0$*, or*
3. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$*.*

Proof : First, we observe that $S \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_2} \to p_{i_2}x_{l_2})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1,\bar{j}_1} + p_{i_2}x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1,\bar{j}_1} + p_{i_2}x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1,\bar{j}_1} + p_{i_2}x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\bar{p}_{i_1,\bar{j}_1}\overline{p_{i_2}x_{l_2}}) + \bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1,\bar{j}_1} + p_{i_2}x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv (p_{i_1}(\overline{p_{i_2}x_{l_2}}) + \bar{p}_{i_1,\bar{j}_1}p_{i_2}\bar{x}_{l_2} + p_{i_1,\bar{j}_1}\bar{p}_{i_1}\bar{p}_{i_2} + 0)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $AB(\overline{A\overline{B}}) \equiv AB$ and $A(\overline{AB}) \equiv A\overline{B}$)

$\equiv p_{i_1}(\overline{p_{i_2}x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+p_{i_2}(\overline{p_{i_1,\bar{j}_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+p_{i_1,\bar{j}_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_1}(\overline{p_{i_2}x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+p_{i_2}(\overline{p_{i_1,\bar{j}_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S}$

Now, $S(\vec{t}) \neq I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_2} \to p_{i_2} x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_2} \to p_{i_2} x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{p_{i_2} x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_2}(\overline{p_{i_1,\bar{j}_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}$ $\cdot\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ such that $p_{i_2} x_{l_2} = 0$,
2. $\vec{t} \in TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ such that $p_{i_1,\bar{j}_1} + x_{l_2} = 0$, or
3. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$.

Hence, the result follows. □

Theorem 4.6 *(LNF with LIF - Case 2)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal $x^{i_1}_{j_1}$ in the i_1-th term, p_{i_1}, in S is negated and the literal x_{l_2} is inserted in p_{i_1}, where $1 \leq i_1 \leq m$, $1 \leq j_1 \leq k_{i_1}$, $k_{i_1}(> 1)$ is the number of literals of p_{i_1}, and x_{l_2} is a missing literal of p_{i_1}, the resulting expression denoted as $I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_1} \to p_{i_1} x_{l_2})}$ is equivalent to that given by double-fault expression (6) in Table 1. Then, we have $S \not\equiv I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_1} \to p_{i_1} x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$, or

2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$.

Proof : First, we observe that $S \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_1} \to p_{i_1} x_{l_2})}$

$\equiv (p_{i_1} \oplus p_{i_1,\bar{j}_1} x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1}(\overline{p_{i_1,\bar{j}_1} x_{l_2}}) + \bar{p}_{i_1}(p_{i_1,\bar{j}_1} x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv (p_{i_1} + p_{i_1,\bar{j}_1}\bar{p}_{i_1}x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

(By making use of $(AB)(\overline{A\overline{B}C}) \equiv AB$)

$\equiv p_{i_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}x_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}x_{l_2}\overline{S}$

Now, $S(\vec{t}) \neq I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_1} \to p_{i_1}x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LIF(p_{i_1} \to p_{i_1}x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}x_{l_2}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$, or

2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$.

Hence, the result follows. □

Theorem 4.7 *(LNF with LRF - Case 1)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal $x^{i_1}_{j_1}$ in the i_1-th term, p_{i_1}, in S is negated and the literal $x^{i_2}_{j_2}$ in the i_2-th term, p_{i_2}, in S is replaced by x_{l_2} where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_1 \leq k_{i_1}$, $1 \leq j_2 \leq k_{i_2}$, $k_{i_1}(> 1)$ and k_{i_2} are the numbers of literals of p_{i_1} and p_{i_2}, respectively, and x_{l_2} is a missing literal of p_{i_2}.

(a) When $k_{i_2} > 1$, the resulting expression denoted as $I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2}x_{l_2})}$ is equivalent to that given by double-fault expression (7) in Table 1. Then, $S \not\equiv I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2}x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in TP_{i_1}(S) \setminus (\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S))$ such that $p_{i_2,\hat{j}_2}x_{l_2} = 0$,

2. $\vec{t} \in TP_{i_2}(S) \setminus (\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S))$ such that $p_{i_1,\bar{j}_1} + x_{l_2} = 0$,

3. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or

4. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$.

(b) When $k_{i_2} = 1$, the resulting expression denoted as $I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}$ is equivalent to double-fault expression (7) in Table 1 without $p_{i_2,\hat{j}_2}$. Then, $S \not\equiv I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ such that $x_{l_2} = 0$,
2. $\vec{t} \in TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ such that $p_{i_1,\bar{j}_1} + x_{l_2} = 0$,
3. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or
4. $\vec{t} \in FP(S)$ such that $x_{l_2} = 1$.

Proof : (a) First, we observe that $S \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}$

$$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1,\bar{j}_1} + p_{i_2,\hat{j}_2} x_{l_2})\big) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1,\bar{j}_1} + p_{i_2,\hat{j}_2} x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1,\bar{j}_1} + p_{i_2,\hat{j}_2} x_{l_2})\big) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2}) \bar{p}_{i_1,\bar{j}_1} (\overline{p_{i_2,\hat{j}_2} x_{l_2}}) + \bar{p}_{i_1} \bar{p}_{i_2} (p_{i_1,\bar{j}_1} + p_{i_2,\hat{j}_2} x_{l_2})\big) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv (p_{i_1} (\overline{p_{i_2,\hat{j}_2} x_{l_2}}) + \bar{p}_{i_1,\bar{j}_1} p_{i_2} \bar{x}_{l_2} + p_{i_1,\bar{j}_1} \bar{p}_{i_1} \bar{p}_{i_2} + p_{i_2,\bar{j}_2} \bar{p}_{i_1} \bar{p}_{i_2} x_{l_2}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$$

(By making use of $AB(\overline{A\bar{B}}) \equiv AB$ and $(AB)(\overline{AC}) \equiv AB\overline{C}$)

$$\equiv p_{i_1} (\overline{p_{i_2,\hat{j}_2} x_{l_2}}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$$
$$+ \bar{p}_{i_1,\bar{j}_1} p_{i_2} \bar{x}_{l_2} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$$
$$+ p_{i_1,\bar{j}_1} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2} \bar{p}_{i_2+1} \cdots \bar{p}_m$$
$$+ p_{i_2,\bar{j}_2} x_{l_2} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2} \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv p_{i_1} (\overline{p_{i_2,\hat{j}_2} x_{l_2}}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$$
$$+ p_{i_2} (\overline{p_{i_1,\bar{j}_1} + x_{l_2}}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1} \overline{S} + p_{i_2,\bar{j}_2} x_{l_2} \overline{S}$$

Now, $S(\vec{t}) \neq I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{p_{i_2,\hat{j}_2} x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+p_{i_2}(\overline{p_{i_1,\bar{j}_1} + x_{l_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S} + p_{i_2,\bar{j}_2} x_{l_2}\overline{S}$

evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S) \big)$ such that $p_{i_2,\hat{j}_2} x_{l_2} = 0$,
2. $\vec{t} \in TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S) \big)$ such that $p_{i_1,\bar{j}_1} + x_{l_2} = 0$,
3. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or
4. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$.

Hence, the result follows.

(b) The proof is similar to part (a) except that the term $p_{i_2,\hat{j}_2}$ does not appear in the proof.

First, we observe that $S \oplus I_{LNF(p_{i_1} \to p_{i_1,\bar{j}_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1,\bar{j}_1} + x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1,\bar{j}_1} + x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1,\bar{j}_1} + x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})\bar{p}_{i_1,\bar{j}_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1,\bar{j}_1} + x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv (p_{i_1}\bar{x}_{l_2} + \bar{p}_{i_1,\bar{j}_1} p_{i_2}\bar{x}_{l_2} + p_{i_1,\bar{j}_1}\bar{p}_{i_1}\bar{p}_{i_2} + \bar{p}_{i_1}\bar{p}_{i_2} x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $AB(\overline{A\overline{B}}) \equiv AB$)

$\equiv p_{i_1}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+\bar{p}_{i_1,\bar{j}_1} p_{i_2}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+p_{i_1,\bar{j}_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+x_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$+p_{i_2}(\overline{p_{i_1,\bar{j}_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m+p_{i_1,\bar{j}_1}\overline{S}+x_{l_2}\overline{S}$

Now, $S(\vec{t}) \neq I_{LNF(p_{i_1}\to p_{i_1,\bar{j}_1})\bowtie LRF(p_{i_2}\to x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LNF(p_{i_1}\to p_{i_1,\bar{j}_1})\bowtie LRF(p_{i_2}\to x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$ $+p_{i_2}(\overline{p_{i_1,\bar{j}_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m+p_{i_1,\bar{j}_1}\overline{S}+x_{l_2}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1,i_2}}^{m} TP_i(S)\big)$ such that $x_{l_2} = 0$,
2. $\vec{t} \in TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1,i_2}}^{m} TP_i(S)\big)$ such that $p_{i_1,\bar{j}_1} + x_{l_2} = 0$,
3. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or
4. $\vec{t} \in FP(S)$ such that $x_{l_2} = 1$.

Hence, the result follows. □

It should be noted that there are two differences between the detection conditions of Theorem 4.7(a) and (b). First, detection condition 1 of Theorem 4.7(a) is related to term $p_{i_2,\hat{j}_2}$ which does not exist in detection condition 1 of Theorem 4.7(b) when $k_{i_2} = 1$ (that is, when p_{i_2} contains just one literal). Second, detection condition 4 of Theorem 4.7(a) (that is, "$\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$") differs from detection condition 4 of Theorem 4.7(b) syntactically. We say that these two conditions are syntactically different because they are actually equivalent to each other when $k_{i_2} = 1$. It is because of the following reason

$\vec{t} \in FP(S)$ such that x_{l_2} = 1

$\equiv$ $\vec{t} \in FP(S)$ such that p_{i_2} = $x_1^{i_2}$ = 0 and x_{l_2} = 1

$\equiv$ $\vec{t} \in FP(S)$ such that $p_{i_2,\bar{1}}$ = $\bar{x}_1^{i_2}$ = 1 and x_{l_2} = 1

$\equiv$ $\vec{t} \in NFP_{i_2,\bar{1}}(S)$ such that x_{l_2} = 1 (Please be noted that j_2 = 1 when k_{i_2} = 1)

Hence, without loss of generality, we can still use the four detection conditions in Theorem 4.7(a) to represent the detection conditions of double-fault expression (7) in Table 1 for both (a) and (b) (that is when $k_{i_2} \geq 1$), bearing in mind that, when $k_{i_2} = 1$, $p_{i_2,\hat{j}_2}x_{l_2}$ degenerates to x_{l_2} and "$\vec{t} \in FP(S)$ such that x_{l_2} = 1" is equivalent to "$\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that x_{l_2} = 1". We will make similar comments in theorems related to special situations when either k_{i_1} or k_{i_2} is equal to 1 or 2 in the sequel.

Theorem 4.8 *(LNF with LRF - Case 2)*

Let $S=p_1+\cdots+p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal $x_{j_1}^{i_1}$ in the i_1-th term, p_{i_1}, in S is negated and the literal $x_{j_2}^{i_1}$ in the i_1-th term, p_{i_1}, in S is replaced by x_{l_2} where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 \leq k_{i_1}$, k_{i_1} is the number of literals of p_{i_1}, and x_{l_2} is a missing literal of p_{i_1}, the resulting expression denoted as $I_{LNF(p_{i_1}\to p_{i_1,\bar{j}_1})\bowtie LRF(p_{i_1}\to p_{i_1,\hat{j}_2}x_{l_2})}$ is equivalent to that given by double-fault expression (8) in Table 1. Then, $S \not\equiv I_{LNF(p_{i_1}\to p_{i_1,\bar{j}_1})\bowtie LRF(p_{i_1}\to p_{i_1,\hat{j}_2}x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S)$, or*
2. *$\vec{t} \in FP(S)$ such that $p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2}$ = 1.*

Proof : First, we observe that $S \oplus I_{LNF(p_{i_1}\to p_{i_1,\bar{j}_1})\bowtie LRF(p_{i_1}\to p_{i_1,\hat{j}_2}x_{l_2})}$

$\equiv (p_{i_1} \oplus p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m$

$\equiv \left(p_{i_1}\overline{p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2}} + \bar{p}_{i_1}p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2}\right)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m$

$\equiv \left(p_{i_1} + \bar{p}_{i_1}p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2}\right)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m$

(By making use of $ABC(\overline{A\bar{B}D}) \equiv ABC$)

$\equiv p_{i_1}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_m$

$\equiv p_{i_1}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2}\overline{S}$

Now,	$S(\vec{t}) \neq I_{LNF(p_{i_1}\to p_{i_1,\bar{j}_1})\bowtie LRF(p_{i_1}\to p_{i_1,\hat{j}_1})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{LNF(p_{i_1}\to p_{i_1,\bar{j}_1})\bowtie LRF(p_{i_1}\to p_{i_1,\hat{j}_2}x_{l_2})}(\vec{t}) = 1$
if and only if	$p_{i_1}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2}\overline{S}$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$, or
2. $\vec{t} \in FP(S)$ such that $p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2}$ = 1.

Hence, the result follows. □

4.2 LOF with other Faults

Theorem 4.9 *(LOF with LOF - Case 1)*

Let S=$p_1+\cdots+p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two literals $x^{i_1}_{j_1}$ in the i_1-th term, p_{i_1}, in S and $x^{i_2}_{j_2}$ in the i_2-th term, p_{i_2}, in S are omitted from p_{i_1} and p_{i_2}, respectively, where $1 \le i_1 < i_2 \le m$, $1 \le j_1 \le k_{i_1}$, $1 \le j_2 \le k_{i_2}$ and $k_{i_1}(>1)$ and $k_{i_2}(>1)$ are the numbers of literals of p_{i_1} and p_{i_2}, respectively, the resulting expression denoted as $I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LOF(p_{i_2}\to p_{i_2,\hat{j}_2})}$ is equivalent to that given by double-fault expression (9) in Table 1. Then, we have $S \not\equiv I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LOF(p_{i_2}\to p_{i_2,\hat{j}_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or*
2. *$\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$.*

Proof : First, we observe that $S \oplus I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LOF(p_{i_2}\to p_{i_2,\hat{j}_2})}$

$\equiv \big((p_{i_1}+p_{i_2})\oplus(p_{i_1,\hat{j}_1}+p_{i_2,\hat{j}_2})\big)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_2})(\overline{p_{i_1,\hat{j}_1}+p_{i_2,\hat{j}_2}})+(\overline{p_{i_1}+p_{i_2}})(p_{i_1,\hat{j}_1}+p_{i_2,\hat{j}_2})\big)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_2})\bar{p}_{i_1,\hat{j}_1}\bar{p}_{i_2,\hat{j}_2}+\bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1,\hat{j}_1}+p_{i_2,\hat{j}_2})\big)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv (0+0+p_{i_1,\bar{j}_1}\bar{p}_{i_1}\bar{p}_{i_2}+\bar{p}_{i_1}\bar{p}_{i_2}p_{i_2,\bar{j}_2})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

(By making use of $AB(\overline{A})\equiv 0$ and rewriting $(\overline{AB})(A)$ as $(\overline{AB})(A\overline{B})$)

becuase they are equivalent)

$\equiv p_{i_1,\bar{j}_1}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$+p_{i_2,\bar{j}_2}\bar{p}_{i_1}\bar{p}_{i_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv p_{i_1,\bar{j}_1}\overline{S}+p_{i_2,\bar{j}_2}\overline{S}$

Now, $S(\vec{t}) \neq I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LOF(p_{i_2}\to p_{i_2,\hat{j}_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LOF(p_{i_2}\to p_{i_2,\hat{j}_2})}(\vec{t}) = 1$

if and only if $p_{i_1,\bar{j}_1}\overline{S}+p_{i_2,\bar{j}_2}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t}\in NFP_{i_1,\bar{j}_1}(S)$, or
2. $\vec{t}\in NFP_{i_2,\bar{j}_2}(S)$.

Hence, the result follows. □

Theorem 4.10 *(LOF with LOF - Case 2)*

Let S=$p_1+\cdots+p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two literals $x^{i_1}_{j_1}$ and $x^{i_1}_{j_2}$ in the i_1-th term, p_{i_1}, in S are omitted from p_{i_1} where $1\leq i_1\leq m$, $1\leq j_1<j_2\leq k_{i_1}$ and $k_{i_1}(>2)$ is the number of literals in p_{i_1}, the resulting expression denoted as $I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LOF(p_{i_1}\to p_{i_1,\hat{j}_2})}$ is equivalent to double-fault expression (10) in Table 1. Then, $S\not\equiv I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LOF(p_{i_1}\to p_{i_1,\hat{j}_2})}$ if and only if there is a test case $\vec{t}\in FP(S)$ such that $p_{i_1,\hat{j}_1,\hat{j}_2}$ = 1.

Proof : First, we observe that $S\oplus I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LOF(p_{i_1}\to p_{i_1,\hat{j}_2})}$

$\equiv (p_{i_1}\oplus p_{i_1,\hat{j}_1,\hat{j}_2})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m$

$\equiv \left(p_{i_1}\bar{p}_{i_1,\hat{j}_1,\hat{j}_2} + \bar{p}_{i_1}p_{i_1,\hat{j}_1,\hat{j}_2}\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1,\hat{j}_1,\hat{j}_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1,\hat{j}_1,\hat{j}_2}\overline{S}$

Now,	$S(\vec{t}) \neq I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LOF(p_{i_1} \to p_{i_1,\hat{j}_1})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LOF(p_{i_1} \to p_{i_1,\hat{j}_2})}(\vec{t}) = 1$
if and only if	$p_{i_1,\hat{j}_1,\hat{j}_2}\overline{S}$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t} \in FP(S)$ such that $p_{i_1,\hat{j}_1,\hat{j}_2} = 1$.

Hence, the result follows. □

Theorem 4.11 *(LOF with LIF - Case 1)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal $x^{i_1}_{j_1}$ in the i_1-th term, p_{i_1}, in S is omitted from p_{i_1} and the literal x_{l_2} is inserted in the i_2-th term, p_{i_2}, in S where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_1 \leq k_{i_1}$, $k_{i_1} (> 1)$ is the number of literals of p_{i_1} and x_{l_2} is a missing literal of p_{i_2}, the resulting expression denoted as $I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LIF(p_{i_2} \to p_{i_2}x_{l_2})}$ is equivalent to that given by double-fault expression (11) in Table 1. Then, $S \not\equiv I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LIF(p_{i_2} \to p_{i_2}x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$, or*
2. *$\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$.*

Proof : First, we observe that $S \oplus I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LIF(p_{i_2} \to p_{i_2}x_{l_2})}$

$\equiv \left((p_{i_1} + p_{i_2}) \oplus (p_{i_1,\hat{j}_1} + p_{i_2}x_{l_2})\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \left((p_{i_1} + p_{i_2})(\overline{p_{i_1,\hat{j}_1} + p_{i_2}x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1,\hat{j}_1} + p_{i_2}x_{l_2})\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \left((p_{i_1} + p_{i_2})\bar{p}_{i_1,\hat{j}_1}(\overline{p_{i_2}x_{l_2}}) + \bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1,\hat{j}_1} + p_{i_2}x_{l_2})\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \left(0 + p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_1,\hat{j}_1} + 0\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $AB(\overline{A}) \equiv 0$ and $A(\overline{AB}) \equiv A\overline{B}$)

$\equiv \left(p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{p}_{i_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_1,\bar{j}_1}\right)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

(By rewriting $(\overline{A})$ as $(\overline{A})(\overline{AB})$ because they are equivalent;

and $(\overline{AB})A$ as $(\overline{AB})A\overline{B}$ because they are equivalent)

$\equiv p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$+p_{i_1,\bar{j}_1}\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv p_{i_2}(\overline{p_{i_1,\hat{j}_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + p_{i_1,\bar{j}_1}\overline{S}$

Now, $S(\vec{t}) \neq I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LIF(p_{i_2}\to p_{i_2}x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LIF(p_{i_2}\to p_{i_2}x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_2}(\overline{p_{i_1,\hat{j}_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + p_{i_1,\hat{j}_1}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$, or

2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$.

Hence, the result follows. □

Theorem 4.12 *(LOF with LRF - Case 1)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal $x^{i_1}_{j_1}$ in the i_1-th term, p_{i_1}, in S is omitted and the literal $x^{i_2}_{j_2}$ in the i_2-th term, p_{i_2}, in S is replaced by x_{l_2} where $1 \le i_1 < i_2 \le m$, $1 \le j_1 \le k_{i_1}$, $k_{i_1}(> 1)$ is the number of literals of p_{i_1} and x_{l_2} is a missing literal of p_{i_2}.

(a) When $k_{i_2} > 1$, the resulting expression denoted as $I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LRF(p_{i_2}\to p_{i_2,\hat{j}_2}x_{l_2})}$ is equivalent to double-fault expression (12) in Table 1. Then, $S \not\equiv I_{LOF(p_{i_1}\to p_{i_1,\hat{j}_1})\bowtie LRF(p_{i_2}\to p_{i_2,\hat{j}_2}x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$,

2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or

3. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$.

(b) When $k_{i_2} = 1$, the resulting expression denoted as $I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \rtimes LRF(p_{i_2} \to x_{l_2})}$ is equivalent to that given by double-fault expression (12) in Table 1. Then, $S \not\equiv I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \rtimes LRF(p_{i_2} \to x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$,
2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or
3. $\vec{t} \in FP(S)$ such that $x_{l_2} = 1$.

Proof : (a) First, we observe that $S \oplus I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \rtimes LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}$

$$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1,\hat{j}_1} + p_{i_2,\hat{j}_2} x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1,\hat{j}_1} + p_{i_2,\hat{j}_2} x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1,\hat{j}_1} + p_{i_2,\hat{j}_2} x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2})\bar{p}_{i_1,\hat{j}_1}\overline{p_{i_2,\hat{j}_2} x_{l_2}} + \bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1,\hat{j}_1} + p_{i_2,\hat{j}_2} x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big(0 + p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_1,\hat{j}_1} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_2,\hat{j}_2} x_{l_2}\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

(By making use of $(AB)(\overline{A}) \equiv 0$ and $(AB)(\overline{AC}) \equiv AB\overline{C}$)

$$\equiv \big(p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_1,\bar{j}_1} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_2,\bar{j}_2} x_{l_2}\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

(By rewriting $(\overline{AB})A$ as $(\overline{AB})A\overline{B}$ becuase they are equivalent)

$$\equiv p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$+ p_{i_1,\bar{j}_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$+ p_{i_2,\bar{j}_2} x_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv p_{i_2}(\overline{p_{i_1,\hat{j}_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S} + p_{i_2,\bar{j}_2} x_{l_2}\overline{S}$$

Now, $S(\vec{t}) \neq I_{LOF(p_{i_1} \rightarrow p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_2} \rightarrow p_{i_2,\hat{j}_2} x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LOF(p_{i_1} \rightarrow p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_2} \rightarrow p_{i_2,\hat{j}_2} x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_2}(\overline{p_{i_1,\hat{j}_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S} + p_{i_2,\bar{j}_2} x_{l_2}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$,
2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or
3. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$.

Hence, the result follows.

(b) The proof is similar to that of (a) above except that the terms $p_{i_2,\hat{j}_2}$ does not appear in the proof.

First, we observe that $S \oplus I_{LOF(p_{i_1} \rightarrow p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_2} \rightarrow x_{l_2})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1,\hat{j}_1} + x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1,\hat{j}_1} + x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1,\hat{j}_1} + x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1,\hat{j}_1} + x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(0 + p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_1,\hat{j}_1} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_2}\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $(AB)(\overline{A}) \equiv 0$)

$\equiv \big(p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_1,\bar{j}_1} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_2}\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

(By rewriting $(\overline{AB})A$ as $(\overline{AB})A\overline{B}$ becuase they are equivalent)

$\equiv p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+p_{i_1,\bar{j}_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+x_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_2}(\overline{p_{i_1,\hat{j}_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S} + x_{l_2}\overline{S}$

Now, $S(\vec{t}) \neq I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_2}(\overline{p_{i_1,\hat{j}_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}\overline{S} + x_{l_2}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$,
2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$, or
3. $\vec{t} \in FP(S)$ such that $x_{l_2} = 1$.

Hence, the result follows. □

It should be noted that there is a difference between detection conditions of Theorem 4.12(a) and (b). Detection condition 3 of Theorem 4.12(a) (that is, "$\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$") differs from detection condition 3 in Theorem 4.12(b) syntactically. We say that these two conditions are syntactically different because they are actually equivalent to each other when $k_{i_2} = 1$ (that is, p_{i_2} contains just one literal). It is because of the following reason

$\vec{t} \in FP(S)$ such that $x_{l_2} = 1$

$\equiv$ $\vec{t} \in FP(S)$ such that $p_{i_2} = x_1^{i_2} = 0$ and $x_{l_2} = 1$

$\equiv$ $\vec{t} \in FP(S)$ such that $p_{i_2,\bar{1}} = \bar{x}_1^{i_2} = 1$ and $x_{l_2} = 1$

$\equiv$ $\vec{t} \in NFP_{i_2,\bar{1}}(S)$ such that $x_{l_2} = 1$ (Please be noted that j_2 = 1 when k_{i_2} = 1)

Hence, without loss of generality, we can still use the three detection conditions in Theorem 4.12(a) to represent the detection conditions of double-fault expression (12) in Table 1 for both situations in Theorem 4.12(a) and (b), bearing in mind the equivalence between "$\vec{t} \in FP(S)$ such that $x_{l_2} = 1$" and "$\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$" when $k_{i_2} = 1$.

Theorem 4.13 *(LOF with LRF - Case 2)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose *that the literal $x^{i_1}_{j_1}$ of the i_1-th term, p_{i_1}, in S is omitted from p_{i_1} and the literal $x^{i_1}_{j_2}$ of the i_1-th term, p_{i_1}, in S is replaced by the literal x_{l_2} where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 \leq k_{i_1}$, k_{i_1} is the number of literals of p_{i_1} and x_{l_2} is a missing literal of p_{i_1}.*

(a) When $k_{i_2} > 2$, the resulting expression denoted as $I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \ltimes LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$ is equivalent to double-fault expression (13) in Table 1. Then, $S \not\equiv I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \ltimes LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_2} = 0$,*
2. *$\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$,*
3. *$\vec{t} \in NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or*
4. *$\vec{t} \in FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} x_{l_2} = 1$.*

(b) When $k_{i_2} = 2$, the resulting expression denoted as $I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \ltimes LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$ is equivalent to that given by double-fault expression (13) in Table 1 without $p_{i_1,\hat{j}_1,\hat{j}_2}$ because p_{i_1} contains just two literals. Then, we have $S \not\equiv I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \ltimes LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_2} = 0$, or*
2. *$\vec{t} \in FP(S)$ such that $x_{l_2} = 1$.*

Proof : (a) First, we observe that $S \oplus I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \ltimes LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$

$\equiv (p_{i_1} \oplus p_{i_1,\hat{j}_1,\hat{j}_2} x_{l_2}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1} \overline{p_{i_1,\hat{j}_1,\hat{j}_2} x_{l_2}} + \bar{p}_{i_1} p_{i_1,\hat{j}_1,\hat{j}_2} x_{l_2}\right) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1} \bar{x}_{l_2} + \bar{p}_{i_1} p_{i_1,\bar{j}_1} x_{l_2} + \bar{p}_{i_1} p_{i_1,\bar{j}_2} x_{l_2} + \bar{p}_{i_1} p_{i_1,\bar{j}_1,\bar{j}_2} x_{l_2}\right) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_m$

(By making use of $(ABC)(\overline{AD}) \equiv ABC\overline{D}$ and $(\overline{ABC})A \equiv (\overline{ABC})(A\overline{B}C + AB\overline{C} + A\overline{BC})$)

$\equiv p_{i_1} \bar{x}_{l_2} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1} x_{l_2} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \bar{p}_{i_1+1} \cdots \bar{p}_m$

$+ p_{i_1,\bar{j}_2} x_{l_2} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1,\bar{j}_2} x_{l_2} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1} \bar{x}_{l_2} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1} x_{l_2} \overline{S} + p_{i_1,\bar{j}_2} x_{l_2} \overline{S} + p_{i_1,\bar{j}_1,\bar{j}_2} x_{l_2} \overline{S}$

Now,	$S(\vec{t}) \neq I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}(\vec{t}) = 1$
if and only if	$p_{i_1}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1} x_{l_2}\overline{S} + p_{i_1,\bar{j}_2} x_{l_2}\overline{S} + p_{i_1,\bar{j}_1,\bar{j}_2} x_{l_2}\overline{S}$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_2} = 0$,
2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$,
3. $\vec{t} \in NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or
4. $\vec{t} \in FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} x_{l_2} = 1$.

Hence, the result follows.

(b) The proof is similar to that of (a) above except that the term $p_{i_1,\hat{j}_1,\hat{j}_2}$ does not appear in the proof.

First, we observe that $S \oplus I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$

$\equiv (p_{i_1} \oplus x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1}\bar{x}_{l_2} + \bar{p}_{i_1} x_{l_2}\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + x_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + x_{l_2}\overline{S}$

Now,	$S(\vec{t}) \neq I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{LOF(p_{i_1} \to p_{i_1,\hat{j}_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}(\vec{t}) = 1$
if and only if	$p_{i_1}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + x_{l_2}\overline{S}$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_2} = 0$, or
2. $\vec{t} \in FP(S)$ such that $x_{l_2} = 1$.

Hence, the result follows. $\square$

Although detection conditions 2, 3 and 4 of Theorem 4.13(a) are syntactically different from the detection condition 2 of Theorem 4.13(b), they are actually equivalent to each other when $k_{i_1} = 2$ (that is p_{i_1} contains just two literals) because of the following reason

$\vec{t} \in FP(S)$ such that $x_{l_2} = 1$

$\equiv$ $\vec{t} \in FP(S)$ such that $p_{i_1} = x_1^{i_1} x_2^{i_1} = 0$ and $x_{l_2} = 1$ (p_{i_1} contains just two literals when $k_{i_1} = 2$)

$\equiv$ $\vec{t} \in FP(S)$ such that $\bar{p}_{i_1} = \overline{x_1^{i_1} x_2^{i_1}} = 1$ and $x_{l_2} = 1$

$\equiv$ $\vec{t} \in FP(S)$ such that $\bar{p}_{i_1} = \bar{x}_1^{i_1} x_2^{i_1} + x_1^{i_1} \bar{x}_2^{i_1} + \bar{x}_1^{i_1} \bar{x}_2^{i_1} = 1$ and $x_{l_2} = 1$

$\equiv$ $\vec{t} \in FP(S)$ such that

(1) $p_{i_1,\bar{1}} = \bar{x}_1^{i_1} x_2^{i_1} = 1$ and $x_{l_2} = 1$,

(2) $p_{i_1,\bar{2}} = x_1^{i_1} \bar{x}_2^{i_1} = 1$ and $x_{l_2} = 1$, or

(3) $p_{i_1,\bar{1},\bar{2}} = \bar{x}_1^{i_1} \bar{x}_2^{i_1} = 1$ and $x_{l_2} = 1$

$\equiv$ $\vec{t} \in FP(S)$ such that

(1) $\vec{t} \in NFP_{i_1,\bar{1}}(S)$ such that $x_{l_2} = 1$,

(2) $\vec{t} \in NFP_{i_1,\bar{2}}(S)$ such that $x_{l_2} = 1$, or

(3) $\vec{t} \in FP(S)$ such that $p_{i_1,\bar{1},\bar{2}} x_{l_2} = 1$

Hence, without loss of generality, we can still use the four detection conditions in Theorem 4.13(a) to represent those of double-fault expression (13) in Table 1 for both (a) and (b), bearing in mind that detection conditions 2, 3 and 4 degenerate to "$\vec{t} \in FP(S)$ such that $x_{l_1} x_{l_2} = 1$" when $k_{i_1} = 2$.

4.3 LIF with other Faults

Theorem 4.14 *(LIF with LIF - Case 1)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two literals x_{l_1} and x_{l_2} are inserted in the i_1-th term, p_{i_1} and the i_2-th term, p_{i_2}, in S, respectively, *where $1 \leq i_1 < i_2 \leq m$, and x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively, the resulting expression denoted as $I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \ltimes LIF(p_{i_2} \to p_{i_2} x_{l_2})}$ is equivalent to that given by double-fault expression (14) in Table 1. Then, we have $S \not\equiv I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \ltimes LIF(p_{i_2} \to p_{i_2} x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:*

1. *$\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} = 0$,*
2. *$\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_2} = 0$, or*
3. *$\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1} + x_{l_2} = 0$.*

Proof : First, we observe that $S \oplus I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \ltimes LIF(p_{i_2} \to p_{i_2} x_{l_2})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1} x_{l_1} + p_{i_2} x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1} x_{l_1} + p_{i_2} x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1} x_{l_1} + p_{i_2} x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})\overline{p_{i_1} x_{l_1}} \cdot \overline{p_{i_2} x_{l_2}} + \bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1} x_{l_1} + p_{i_2} x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1}\bar{x}_{l_1}\overline{p_{i_2} x_{l_2}} + p_{i_2}\bar{x}_{l_2}\overline{p_{i_1} x_{l_1}} + 0 + 0\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $A(\overline{AB}) \equiv A\overline{B}$)

$\equiv \big(p_{i_1}\bar{x}_{l_1}(\bar{p}_{i_2} + p_{i_2}\bar{x}_{l_2}) + p_{i_2}\bar{x}_{l_2}(\bar{p}_{i_1} + p_{i_1}\bar{x}_{l_1})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

(By rewriting $\overline{AB}$ as $\overline{A} + A \cdot \overline{B}$ because they are equivalent)

$\equiv \big(p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1} + p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}p_{i_2}\bar{x}_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_2}\bar{x}_{l_1}\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1} + \bar{p}_{i_1}p_{i_2}\bar{x}_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_2}\bar{x}_{l_1}\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{x}_{l_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2} \cdots \bar{p}_m + p_{i_2}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+ p_{i_1}p_{i_2}\bar{x}_{l_2}\bar{x}_{l_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

Now, $S(\vec{t}) \neq I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \bowtie LIF(p_{i_2} \to p_{i_2} x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \bowtie LIF(p_{i_2} \to p_{i_2} x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}\bar{x}_{l_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2} \cdots \bar{p}_m \ + \ p_{i_2}\bar{x}_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m \ +$
$p_{i_1}p_{i_2}\bar{x}_{l_2}\bar{x}_{l_1}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} = 0$,
2. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_2} = 0$, or
3. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1} + x_{l_2} = 0$.

Hence, the result follows. □

Theorem 4.15 *(LIF with LIF - Case 2)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two literals x_{l_1} and x_{l_2} from different Boolean variables are inserted in the i_1-th term, p_{i_1}, in S where $1 \leq i_1 \leq m$ and x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}, the resulting expression denoted as $I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \bowtie LIF(p_{i_1} \to p_{i_1} x_{l_2})}$ is equivalent to double-fault expression (15) in Table 1. Then, $S \not\equiv I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \bowtie LIF(p_{i_1} \to p_{i_1} x_{l_2})}$ if and only if there is a test case $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1}x_{l_2} = 0$.

Proof : First, we observe that $S \oplus I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \bowtie LIF(p_{i_1} \to p_{i_1} x_{l_2})}$

$\equiv (p_{i_1} \oplus p_{i_1}x_{l_1}x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1}\overline{p_{i_1}x_{l_1}x_{l_2}} + \bar{p}_{i_1}(p_{i_1}x_{l_1}x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1}\overline{x_{l_1}x_{l_2}} + 0\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

(By making use of $A(\overline{ABC}) \equiv A(\overline{BC})$)

$\equiv p_{i_1}\overline{x_{l_1}x_{l_2}}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

Now, $S(\vec{t}) \neq I_{LIF(p_{i_1} \rightarrow p_{i_1} x_{l_1}) \ltimes LIF(p_{i_1} \rightarrow p_{i_1} x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LIF(p_{i_1} \rightarrow p_{i_1} x_{l_1}) \ltimes LIF(p_{i_1} \rightarrow p_{i_1} x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1} \overline{x_{l_1} x_{l_2}} \bar{p}_1 \cdots \bar{p}_{i_1 - 1} \bar{p}_{i_1 + 1} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} x_{l_2} = 0$.

Hence, the result follows. □

Theorem 4.16 *(LIF with LRF - Case 1)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal x_{l_1} is inserted in the i_1-th term, p_{i_1}, in S and the literal $x^{i_2}_{j_2}$ in the i_2-th term, p_{i_2}, in S is replaced by x_{l_2} where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_2 \leq k_{i_2}$, k_{i_2} is the number of literals of p_{i_2}, and x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively.

(a) When $k_{i_2} > 1$, the resulting expression denoted as $I_{LIF(p_{i_1} \rightarrow p_{i_1} x_{l_1}) \ltimes LRF(p_{i_2} \rightarrow \ p_{i_2, \hat{j}_2} x_{l_2})}$ is equivalent to double-fault expression (16) in Table 1. Then, $S \not\equiv I_{LIF(p_{i_1} \rightarrow p_{i_1} x_{l_1}) \ltimes LRF(p_{i_2} \rightarrow p_{i_2, \hat{j}_2} x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2, \hat{j}_2} + x_{l_1} = 0$,*
2. *$\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$,*
3. *$\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_2} = 0$,*
4. *$\vec{t} \in NFP_{i_2, \bar{j}_2}(S)$ such that $x_{l_2} = 1$, or*
5. *$\vec{t} \in (TP_{i_1}(S) \cap TP_{i_2}(S)) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ such that $x_{l_1} + x_{l_2} = 0$.*

(b) When $k_{i_2} = 1$, the resulting expression denoted as $I_{LIF(p_{i_1} \rightarrow p_{i_1} x_{l_1}) \ltimes LRF(p_{i_2} \rightarrow x_{l_2})}$ is equivalent to that given by double-fault expression (16) in Table 1 without $p_{i_2, \hat{j}_2}$ because p_{i_2} contains just one literal. Then, we have $S \not\equiv I_{LIF(p_{i_1} \rightarrow p_{i_1} x_{l_1}) \ltimes LRF(p_{i_2} \rightarrow x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$,*

2. $\vec{t} \in UTP_{i_2}(S)$ *such that* $x_{l_2} = 0$,

3. $\vec{t} \in FP(S)$ *such that* $x_{l_2} = 1$, *or*

4. $\vec{t} \in (TP_{i_1}(S) \cap TP_{i_2}(S)) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ *such that* $x_{l_1} + x_{l_2} = 0$.

Proof : (a) First, we observe that $S \oplus I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2, \hat{j}_2} x_{l_2})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1} x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2})\big) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1} x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1} x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2})\big) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2}) \overline{p_{i_1} x_{l_1}} \cdot \overline{p_{i_2,\hat{j}_2} x_{l_2}} + \bar{p}_{i_1} \bar{p}_{i_2} (p_{i_1} x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2})\big) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1} \bar{x}_{l_1} \overline{p_{i_2,\hat{j}_2} x_{l_2}} + p_{i_2} \bar{x}_{l_2} \overline{p_{i_1} x_{l_1}} + 0 + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_2,\hat{j}_2} x_{l_2}\big) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $A(\overline{AB}) \equiv A\overline{B}$ and $AB(\overline{AC}) \equiv AB\overline{C}$)

$\equiv \big(p_{i_1} \bar{x}_{l_1} (\bar{p}_{i_2,\hat{j}_2} \bar{p}_{i_2} + \bar{p}_{i_2} \bar{x}_{l_2} + p_{i_2} \bar{x}_{l_2}) + p_{i_2} \bar{x}_{l_2} (\bar{p}_{i_1} + p_{i_1} \bar{x}_{l_1}) + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_2,\bar{j}_2} x_{l_2}\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $\overline{AC} \equiv \overline{A} \cdot (\overline{AB}) + (\overline{AB}) \cdot \overline{C} + AB\overline{C}$, $\overline{AB} \equiv \overline{A} + A \cdot \overline{B}$ and $\overline{ABA} \equiv \overline{AB} \cdot A\overline{B}$)

$\equiv \big(p_{i_1} \bar{x}_{l_1} \bar{p}_{i_2,\hat{j}_2} \bar{p}_{i_2} + p_{i_1} \bar{p}_{i_2} \bar{x}_{l_1} \bar{x}_{l_2} + p_{i_1} p_{i_2} \bar{x}_{l_1} \bar{x}_{l_2} + \bar{p}_{i_1} p_{i_2} \bar{x}_{l_2} + p_{i_1} p_{i_2} \bar{x}_{l_1} \bar{x}_{l_2} + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_2,\bar{j}_2} x_{l_2}\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1} \bar{p}_{i_2,\hat{j}_2} \bar{x}_{l_1} \bar{p}_{i_2} + p_{i_1} \bar{p}_{i_2} \bar{x}_{l_1} \bar{x}_{l_2} + \bar{p}_{i_1} p_{i_2} \bar{x}_{l_2} + p_{i_1} p_{i_2} \bar{x}_{l_1} \bar{x}_{l_2} + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_2,\bar{j}_2} x_{l_2}\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1} \bar{p}_{i_2,\hat{j}_2} \bar{x}_{l_1} \bar{p}_{i_2} + p_{i_1} \bar{p}_{i_2} \bar{x}_{l_1} \bar{x}_{l_2} + \bar{p}_{i_1} p_{i_2} \bar{x}_{l_2} + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_2,\bar{j}_2} x_{l_2} + p_{i_1} p_{i_2} \bar{x}_{l_1} \bar{x}_{l_2}\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1} (\overline{p_{i_2,\hat{j}_2} + x_{l_1}}) \bar{p}_{i_2} + p_{i_1} \bar{p}_{i_2} (\overline{x_{l_1} + x_{l_2}}) + \bar{p}_{i_1} p_{i_2} \bar{x}_{l_2} + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_2,\bar{j}_2} x_{l_2} + p_{i_1} p_{i_2} (\overline{x_{l_1} + x_{l_2}})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_1} (\overline{p_{i_2,\hat{j}_2} + x_{l_1}}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$+ p_{i_1} (\overline{x_{l_1} + x_{l_2}}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$+ p_{i_2} \bar{x}_{l_2} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$+ \bar{p}_{i_2,\bar{j}_2} x_{l_2} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$+ p_{i_1} p_{i_2} (\overline{x_{l_1} + x_{l_2}}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_1} (\overline{p_{i_2,\hat{j}_2} + x_{l_1}}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1} (\overline{x_{l_1} + x_{l_2}}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_m$

$$+p_{i_2}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m+\bar{p}_{i_2,\bar{j}_2}x_{l_2}\overline{S}+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

Now, $S(\vec{t}) \neq I_{LIF(p_{i_1}\to p_{i_1}x_{l_1})\bowtie LRF(p_{i_2}\to p_{i_2,\hat{j}_2}x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LIF(p_{i_1}\to p_{i_1}x_{l_1})\bowtie LRF(p_{i_2}\to p_{i_2,\hat{j}_2}x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{p_{i_2,\hat{j}_2}+x_{l_1}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m$
$+p_{i_2}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m+\bar{p}_{i_2,\bar{j}_2}x_{l_2}\overline{S}+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}$
$\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2}+x_{l_1} = 0$,
2. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1}+x_{l_2} = 0$,
3. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_2} = 0$,
4. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or
5. $\vec{t} \in (TP_{i_1}(S)\cap TP_{i_2}(S)) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1}+x_{l_2} = 0$.

Hence, the result follows.

(b) The proof is similar to that in (a) except that the term $p_{i_2,\hat{j}_2}$ does not appear in the proof.

First, we observe that $S \oplus I_{LIF(p_{i_1}\to p_{i_1}x_{l_1})\bowtie LRF(p_{i_2}\to x_{l_2})}$

$\equiv \big((p_{i_1}+p_{i_2})\oplus(p_{i_1}x_{l_1}+x_{l_2})\big)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_2})(\overline{p_{i_1}x_{l_1}+x_{l_2}})+(\overline{p_{i_1}+p_{i_2}})(p_{i_1}x_{l_1}+x_{l_2})\big)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_2})\overline{p_{i_1}x_{l_1}}\cdot\overline{x_{l_2}}+\bar{p}_{i_1}\bar{p}_{i_2}(p_{i_1}x_{l_1}+x_{l_2})\big)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv \big(p_{i_1}\bar{x}_{l_1}\overline{x_{l_2}}+p_{i_2}\bar{x}_{l_2}\overline{p_{i_1}x_{l_1}}+0+\bar{p}_{i_1}\bar{p}_{i_2}x_{l_2}\big)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

(By making use of $A(\overline{AB}) \equiv A\overline{B}$ and $AB(\overline{AC}) \equiv AB\overline{C}$)

$\equiv \big(p_{i_1}\bar{x}_{l_1}(\bar{p}_{i_2}\bar{x}_{l_2}+p_{i_2}\bar{x}_{l_2})+p_{i_2}\bar{x}_{l_2}(\bar{p}_{i_1}+p_{i_1}\bar{x}_{l_1})+\bar{p}_{i_1}\bar{p}_{i_2}x_{l_2}\big)$

$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$

(By making use of $\overline{AC} \equiv \overline{A}\cdot(\overline{AB})+(\overline{AB})\cdot\overline{C}+AB\overline{C}$, $\overline{AB} \equiv \overline{A}+A\cdot\overline{B}$ and $\overline{AB}A \equiv \overline{AB}\cdot A\overline{B}$)

$$\equiv \big(p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}p_{i_2}\bar{x}_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_2}\big)$$

$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv \big(p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}p_{i_2}\bar{x}_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_2}\big)\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv \big(p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}p_{i_2}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}\big)\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv \big(p_{i_1}\bar{p}_{i_2}(\overline{x_{l_1}+x_{l_2}}) + \bar{p}_{i_1}p_{i_2}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_2} + p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\big)\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_2}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + x_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_2}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + x_{l_2}\overline{S}$$

$$+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

Now, $S(\vec{t}) \neq I_{LIF(p_{i_1}\to p_{i_1}x_{l_1})\bowtie LRF(p_{i_2}\to x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LIF(p_{i_1}\to p_{i_1}x_{l_1})\bowtie LRF(p_{i_2}\to x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_2}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + x_{l_2}\overline{S} +$ $p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1}+x_{l_2} = 0$,
2. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_2} = 0$,
3. $\vec{t} \in FP(S)$ such that $x_{l_2} = 1$, or
4. $\vec{t} \in (TP_{i_1}(S) \cap TP_{i_2}(S)) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1}+x_{l_2} = 0$.

Hence, the result follows. □

It should be noted that there are two differences between detection conditions of Theorem 4.16(a) and (b). First, detection condition 1 of Theorem 4.16(a) is related to term $p_{i_2,\hat{j}_2}$ which does not exist in Theorem 4.16(b) when $k_{i_2} = 1$ (that is, when p_{i_2} contains just one literal). Second, detection condition 4 of Theorem 4.16(a) (that is, "$\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$") differs from detection condition 3 in Theorem 4.16(b) syntactically. We say that these two conditions are syntactically different because they are actually equivalent to each other when $k_{i_2} = 1$. It is because

$$\begin{aligned} & \vec{t} \in FP(S) \text{ such that } x_{l_2} = 1 \\ \equiv \quad & \vec{t} \in FP(S) \text{ such that } p_{i_2} = x_1^{i_2} = 0 \text{ and } x_{l_2} = 1 \\ \equiv \quad & \vec{t} \in FP(S) \text{ such that } p_{i_2,\bar{1}} = \bar{x}_1^{i_2} = 1 \text{ and } x_{l_2} = 1 \\ \equiv \quad & \vec{t} \in NFP_{i_2,\bar{1}}(S) \text{ such that } x_{l_2} = 1 \qquad \text{(Please be noted that } j_2 = 1 \text{ when } k_{i_2} = 1) \end{aligned}$$

Hence, without loss of generality, we can still use the five detection conditions in Theorem 4.16(a) to represent those of double-fault expression (16) in Table 1 for both situations in Theorem 4.16(a) and (b), bearing in mind the non-existence of the detection condition 1 of (a) and the equivalence between "$\vec{t} \in FP(S)$ such that $x_{l_2} = 1$" and "$\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$" when $k_{i_2} = 1$.

Theorem 4.17 *(LIF with LRF - Case 2)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal x_{l_1} is inserted in the i_1-th term, p_{i_1}, in S and the literal $x_{j_2}^{i_1}$ in p_{i_1} is replaced by x_{l_2} where $1 \le i_1 \le m$, $1 \le j_2 \le k_{i_1}$, k_{i_1} is the number of literals in p_{i_1} and x_{l_1} and x_{l_2} are two different missing literals of p_{i_1} from different Boolean variables (that is, $x_{l_1} \ne x_{l_2}$ and $x_{l_1} \ne \bar{x}_{l_2}$).

(a) When $k_{i_1} > 1$, the resulting expression denoted as $I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \ltimes LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$ is equivalent to that given by double-fault expression (17) in Table 1. Then, $S \not\equiv I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \ltimes LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} x_{l_2} = 0$, or*
2. *$\vec{t} \in NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1} x_{l_2} = 1$.*

(b) When $k_{i_1} = 1$, the resulting expression denoted as $I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \ltimes LRF(p_{i_1} \to x_{l_2})}$ is equivalent to that given by double-fault expression (17) in Table 1 without $p_{i_1,\hat{j}_1}$ because p_{i_1} contains just one literal. Then, $S \not\equiv I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \ltimes LRF(p_{i_1} \to x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1}x_{l_2} = 0$, or

2. $\vec{t} \in FP(S)$ such that $x_{l_1}x_{l_2} = 1$.

Proof : (a) First, we observe that $S \oplus I_{LIF(p_{i_1} \to p_{i_1}x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2}x_{l_1})}$

$\equiv (p_{i_1} \oplus p_{i_1,\hat{j}_2}x_{l_1}x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1}\overline{p_{i_1,\hat{j}_2}x_{l_1}x_{l_2}} + \bar{p}_{i_1}p_{i_1,\hat{j}_2}x_{l_1}x_{l_2}\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1}\overline{x_{l_1}x_{l_2}} + \bar{p}_{i_1}p_{i_1,\bar{j}_2}x_{l_1}x_{l_2}\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

(By making use of $AB(\overline{ACD}) \equiv AB(\overline{CD})$ and $\overline{AB}(A) \equiv \overline{AB}(A\bar{B})$)

$\equiv p_{i_1}\overline{x_{l_1}x_{l_2}}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_2}x_{l_1}x_{l_2}\overline{S}$

Now, $S(\vec{t}) \neq I_{LIF(p_{i_1} \to p_{i_1}x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2}x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LIF(p_{i_1} \to p_{i_1}x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2}x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}\overline{x_{l_1}x_{l_2}}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_2}x_{l_1}x_{l_2}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1}x_{l_2} = 0$, or

2. $\vec{t} \in NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1}x_{l_2} = 1$.

Hence, the result follows.

(b) The proof is similar to that in (a) except that the term $p_{i_1,\hat{j}_2}$ does not appear in the proof.

First, we observe that $S \oplus I_{LIF(p_{i_1} \to p_{i_1}x_{l_1}) \bowtie LRF(p_{i_1} \to x_{l_1})}$

$\equiv (p_{i_1} \oplus x_{l_1}x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1}\overline{x_{l_1}x_{l_2}} + \bar{p}_{i_1}x_{l_1}x_{l_2}\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\overline{x_{l_1}x_{l_2}}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_2}x_{l_1}x_{l_2}\overline{S}$

Now, $S(\vec{t}) \neq I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \bowtie LRF(p_{i_1} \to x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LIF(p_{i_1} \to p_{i_1} x_{l_1}) \bowtie LRF(p_{i_1} \to x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1} \overline{x_{l_1}} \overline{x_{l_2}} \bar{p}_1 \cdots \bar{p}_{i_1 - 1} \bar{p}_{i_1+1} \cdots \bar{p}_m + x_{l_1} x_{l_2} \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} x_{l_2} = 0$, or
2. $\vec{t} \in FP(S)$ such that $x_{l_1} x_{l_2} = 1$.

Hence, the result follows. □

It should be noted that detection condition 2 in Theorem 4.17(a) (that is, "$\vec{t} \in NFP_{i_1, \bar{j}_2}(S)$ such that $x_{l_1} x_{l_2} = 1$") is equivalent to that in Theorem 4.17(b) (that is, "$\vec{t} \in FP(S)$ such that $x_{l_1} x_{l_2} = 1$"). The reason is similar to those given in the paragraph after Theorem 4.7. Hence, without loss of generality, we can still use the detection conditions in Theorem 4.17(a) to represent the detection conditions of double-fault expression (17) in Table 1 for $k_{i_1} \geq 1$, bearing in mind the equivalence between "$\vec{t} \in FP(S)$ such that $x_{l_1} x_{l_2}$ = 1" and "$\vec{t} \in NFP_{i_1, \bar{j}_2}(S)$ such that $x_{l_1} x_{l_2}$ = 1" when $k_{i_i} = 1$.

4.4 LRF with other Faults

Theorem 4.18 *(LRF with LRF - Case 1)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the literal $x^{i_1}_{j_1}$ in the i_1-th term, p_{i_1}, is replaced by x_{l_1} and the literal $x^{i_2}_{j_2}$ in the i_2-th term, p_{i_2}, is replaced by x_{l_2} where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_1 \leq k_{i_1}$, $1 \leq j_2 \leq k_{i_2}$, k_{i_1} and k_{i_2} are the numbers of literals of p_{i_2} and p_{i_2}, respectively, and x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively.

(a) When $k_{i_1}, k_{i_2} > 1$, the resulting expression denoted as $I_{LRF(p_{i_1} \to p_{i_1, \hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2, \hat{j}_2} x_{l_2})}$ is equivalent to double-fault expression (18) in Table 1. Then, $S \not\equiv I_{LRF(p_{i_1} \to p_{i_1, \hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2, \hat{j}_2} x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$,

2. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

3. $\vec{t} \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$,

4. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

5. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1} = 1$,

6. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or

7. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1} + x_{l_2} = 0$.

(b) When k_{i_1} = 1 and $k_{i_2} > 1$, the resulting expression denoted as $I_{LRF(p_{i_1} \to x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}$ is equivalent to that given by double-fault expression (18) in Table 1 without $p_{i_1,\hat{j}_1}$ because p_{i_1} contains just one literal. Then, $S \not\equiv I_{LRF(p_{i_1} \to x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$,

2. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

3. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

4. $\vec{t} \in FP(S)$ such that $x_{l_1} = 1$,

5. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or

6. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1} + x_{l_2} = 0$.

(c) When $k_{i_1} > 1$ and $k_{i_2} = 1$, the resulting expression denoted as $I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}$ is equivalent to that given by double-fault expression (18) in Table 1 without $p_{i_2,\hat{j}_2}$ because p_{i_2} contains just one literal. Then, we have $S \not\equiv I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

2. $\vec{t} \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$,

3. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

4. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1} = 1$,

5. $\vec{t} \in FP(S)$ such that $x_{l_2} = 1$, or

6. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1} + x_{l_2} = 0$.

(d) When $k_{i_1} = k_{i_2} = 1$, the resulting expression denoted as $I_{LRF(p_{i_1} \to x_{l_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}$ is equivalent to that given by double-fault expression (18) in Table 1 without $p_{i_1,\hat{j}_1}$ and $p_{i_2,\hat{j}_2}$ because both p_{i_1} and p_{i_2} contain just 1 literal. Then, we have $S \not\equiv I_{LRF(p_{i_1} \to x_{l_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

2. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

3. $\vec{t} \in FP(S)$ such that $x_{l_1} = 1$,

4. $\vec{t} \in FP(S)$ such that $x_{l_2} = 1$, or

5. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1} + x_{l_2} = 0$.

Proof : (a) First, we observe that $S \oplus I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1,\hat{j}_1} x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2})\big) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1,\hat{j}_1} x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(p_{i_1,\hat{j}_1} x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2}) \overline{p_{i_1,\hat{j}_1} x_{l_1}} \cdot \overline{p_{i_2,\hat{j}_2} x_{l_2}} + \bar{p}_{i_1} \bar{p}_{i_2} (p_{i_1,\hat{j}_1} x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1} \bar{x}_{l_1} (\overline{p_{i_2,\hat{j}_2} x_{l_2}}) + p_{i_2} \bar{x}_{l_2} (\overline{p_{i_1,\hat{j}_1} x_{l_1}}) + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_1,\hat{j}_1} x_{l_1} + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_2,\hat{j}_2} x_{l_2}\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$ (By making use of $AB(\overline{AC}) \equiv AB\overline{C}$)

$\equiv \big(p_{i_1} \bar{x}_{l_1} (\bar{p}_{i_2,\hat{j}_2} \bar{p}_{i_2} + \bar{p}_{i_2} \bar{x}_{l_2} + p_{i_2} \bar{x}_{l_2}) + p_{i_2} \bar{x}_{l_2} (\bar{p}_{i_1,\hat{j}_1} \bar{p}_{i_1} + \bar{p}_{i_1} \bar{x}_{l_1} + p_{i_1} \bar{x}_{l_1}) + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_1,\bar{j}_1} x_{l_1} + \bar{p}_{i_1} \bar{p}_{i_2} p_{i_2,\bar{j}_2} x_{l_2}\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $\overline{AC} \equiv \overline{A} \cdot (\overline{AB}) + (\overline{AB}) \cdot \overline{C} + AB\overline{C}$ and $\overline{AB} \cdot A \equiv A\bar{B}$)

$$\equiv \big(p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{p}_{i_2}\bar{x}_{l_1} + p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1,\hat{j}_1}\bar{p}_{i_1}p_{i_2}\bar{x}_{l_2} + \bar{p}_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}$$

$$+\bar{p}_{i_1}\bar{p}_{i_2}p_{i_1,\bar{j}_1}x_{l_1} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_2,\bar{j}_2}x_{l_2}\big)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv \big(p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{p}_{i_2}\bar{x}_{l_1} + p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1,\hat{j}_1}\bar{p}_{i_1}p_{i_2}\bar{x}_{l_2} + \bar{p}_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}$$

$$+\bar{p}_{i_1}\bar{p}_{i_2}p_{i_1,\bar{j}_1}x_{l_1} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_2,\bar{j}_2}x_{l_2}\big)\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{x}_{l_1}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1}\bar{x}_{l_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_2}\bar{p}_{i_1,\hat{j}_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1,\bar{j}_1}x_{l_1}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_2,\bar{j}_2}x_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv p_{i_1}(\overline{p_{i_2,\hat{j}_2}+x_{l_1}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m$$

$$+p_{i_2}(\overline{p_{i_1,\hat{j}_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + p_{i_2}(\overline{x_{l_2}+x_{l_1}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + p_{i_1,\bar{j}_1}x_{l_1}\overline{S} + p_{i_2,\bar{j}_2}x_{l_2}\overline{S}$$

$$\equiv p_{i_1}(\overline{p_{i_2,\hat{j}_2}+x_{l_1}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m$$

$$+p_{i_2}(\overline{p_{i_1,\hat{j}_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + p_{i_2}(\overline{x_{l_2}+x_{l_1}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1,\bar{j}_1}x_{l_1}\overline{S} + p_{i_2,\bar{j}_2}x_{l_2}\overline{S} + p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

Now,	$S(\vec{t}) \neq I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}(\vec{t}) = 1$
if and only if	$p_{i_1}(\overline{p_{i_2,\hat{j}_2} + x_{l_1}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1}(\overline{x_{l_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$
	$+p_{i_2}(\overline{p_{i_1,\hat{j}_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + p_{i_2}(\overline{x_{l_2} + x_{l_1}})\bar{p}_1 \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$
	$+p_{i_1,\bar{j}_1} x_{l_1}\overline{S} + p_{i_2,\bar{j}_2} x_{l_2}\overline{S} + p_{i_1}p_{i_2}(\overline{x_{l_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$
	evaluates to 1 on $\vec{t}$
if and only if	$\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$, or
2. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$, or
3. $\vec{t} \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$, or
4. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_1} + x_{l_2} = 0$, or
5. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1} = 1$, or
6. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or
7. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1} + x_{l_2} = 0$.

Hence, the result follows.

(b) The proof is similar to that of (a) above except that the term $p_{i_1,\hat{j}_1}$ does not appear in the proof.

First, we observe that $S \oplus I_{LRF(p_{i_1} \to x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}$

$$\equiv \big((p_{i_1} + p_{i_2}) \oplus (x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2})(\overline{x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2})\bar{x}_{l_1} \cdot \overline{p_{i_2,\hat{j}_2} x_{l_2}} + \bar{p}_{i_1}\bar{p}_{i_2}(x_{l_1} + p_{i_2,\hat{j}_2} x_{l_2})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big(p_{i_1}\bar{x}_{l_1}(\overline{p_{i_2,\hat{j}_2} x_{l_2}}) + p_{i_2}\bar{x}_{l_2}\bar{x}_{l_1} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_1} + \bar{p}_{i_1}\bar{p}_{i_2}p_{i_2,\hat{j}_2} x_{l_2}\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$$

(By making use of $AB(\overline{AC}) \equiv AB\overline{C}$)

$$\equiv \big(p_{i_1}\bar{x}_{l_1}(\bar{p}_{i_2,\hat{j}_2}\bar{p}_{i_2}+\bar{p}_{i_2}\bar{x}_{l_2}+p_{i_2}\bar{x}_{l_2})+p_{i_2}\bar{x}_{l_2}(\bar{p}_{i_1}\bar{x}_{l_1}+p_{i_1}\bar{x}_{l_1})+\bar{p}_{i_1}\bar{p}_{i_2}x_{l_1}+\bar{p}_{i_1}\bar{p}_{i_2}p_{i_2,\bar{j}_2}x_{l_2}\big)$$

$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

(By making use of $\overline{AC}\equiv\overline{A}\cdot(\overline{AB})+(\overline{AB})\cdot\overline{C}+AB\overline{C}$ and $\overline{AB}\cdot A\equiv A\bar{B}$)

$$\equiv \big(p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{p}_{i_2}\bar{x}_{l_1}+p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}+p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}+\bar{p}_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}+p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}+\bar{p}_{i_1}\bar{p}_{i_2}x_{l_1}+\bar{p}_{i_1}\bar{p}_{i_2}p_{i_2,\bar{j}_2}x_{l_2}\big)$$

$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv \big(p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{p}_{i_2}\bar{x}_{l_1}+p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}+\bar{p}_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}+p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}+\bar{p}_{i_1}\bar{p}_{i_2}x_{l_1}+\bar{p}_{i_1}\bar{p}_{i_2}p_{i_2,\bar{j}_2}x_{l_2}\big)$$

$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv p_{i_1}\bar{p}_{i_2,\hat{j}_2}\bar{x}_{l_1}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1}\bar{x}_{l_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+x_{l_1}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_2,\bar{j}_2}x_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv p_{i_1}(\overline{p_{i_2,\hat{j}_2}+x_{l_1}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m+p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m$$

$$+p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+x_{l_1}\overline{S}+p_{i_2,\bar{j}_2}x_{l_2}\overline{S}$$

$$\equiv p_{i_1}(\overline{p_{i_2,\hat{j}_2}+x_{l_1}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m+p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m$$

$$+p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m+x_{l_1}\overline{S}+p_{i_2,\bar{j}_2}x_{l_2}\overline{S}$$

$$+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

Now, $S(\vec{t}) \neq I_{LRF(p_{i_1} \to x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LRF(p_{i_1} \to x_{l_1}) \bowtie LRF(p_{i_2} \to p_{i_2,\hat{j}_2} x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{p_{i_2,\hat{j}_2} + x_{l_1}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1}(\overline{x_{l_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$+p_{i_2}(\overline{x_{l_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m + x_{l_1}\overline{S} + p_{i_2,\bar{j}_2} x_{l_2}\overline{S}$

$+p_{i_1}p_{i_2}(\overline{x_{l_1} + x_{l_2}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$, or
2. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$, or
3. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_1} + x_{l_2} = 0$, or
4. $\vec{t} \in FP(S)$ such that $x_{l_1} = 1$, or
5. $\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or
6. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1} + x_{l_2} = 0$.

Hence, the result follows.

(c) The proof is similar to that of (b) except that $p_{i_2\hat{j}_2}$ does not appear in the proof instead of $p_{i_1,\hat{j}_1}$. We will omit the proof here.

(d) The proof is similar to that of (a) above except that both $p_{i_1,\hat{j}_1}$ and $p_{i_2\hat{j}_2}$ do not appear in the proof. We proceed the proof as follows.

First, we observe that $S \oplus I_{LRF(p_{i_1} \to x_{l_1}) \bowtie LRF(p_{i_2} \to x_{l_2})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (x_{l_1} + x_{l_2})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{x_{l_1} + x_{l_2}}) + (\overline{p_{i_1} + p_{i_2}})(x_{l_1} + x_{l_2})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}(x_{l_1} + x_{l_2})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1}\bar{x}_{l_1}\bar{x}_{l_2} + p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_1} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_2}\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$$\equiv \big(p_{i_1}\bar{p}_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_1} + \bar{p}_{i_1}\bar{p}_{i_2}x_{l_2}\big)$$

$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

(By rewriting $A\bar{C}\bar{D}+B\bar{C}\bar{D}$ as $A\bar{B}\bar{C}\bar{D}+\bar{A}B\bar{C}\bar{D}+AB\bar{C}\bar{D}$ because they are equivalent)

$$\equiv p_{i_1}\bar{x}_{l_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1}p_{i_2}\bar{x}_{l_1}\bar{x}_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+x_{l_1}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+x_{l_2}\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$\equiv p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

$$+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + x_{l_1}\overline{S}+x_{l_2}\overline{S}$$

$$\equiv p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + x_{l_1}\overline{S}+x_{l_2}\overline{S}$$

$$+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$$

Now, $S(\vec{t}) \neq I_{LRF(p_{i_1}\to x_{l_1})\bowtie LRF(p_{i_2}\to x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LRF(p_{i_1}\to x_{l_1})\bowtie LRF(p_{i_2}\to x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_m + p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m + x_{l_1}\overline{S}$ $+x_{l_2}\overline{S}+p_{i_1}p_{i_2}(\overline{x_{l_1}+x_{l_2}})\bar{p}_1\cdots\bar{p}_{i_1-1}\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2+1}\cdots\bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1}+x_{l_2}=0$, or
2. $\vec{t} \in UTP_{i_2}(S)$ such that $x_{l_1}+x_{l_2}=0$, or
3. $\vec{t} \in FP(S)$ such that $x_{l_1}=1$, or
4. $\vec{t} \in FP(S)$ such that $x_{l_2}=1$, or
5. $\vec{t} \in \big(TP_{i_1}(S)\cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{i=1, i\neq i_1,i_2}^{m} TP_i(S)\big)$ such that $x_{l_1}+x_{l_2}=0$.

Hence, the result follows. □

It should be noted that there are two differences between detection conditions of Theorem 4.18(a) and (b). First, detection condition 3 of Theorem 4.18(a) is related to term $p_{i_1,\hat{j}_1}$ which does not exist in Theorem 4.18(b) when $k_{i_1} = 1$ (that is, when p_{i_1} contains just one literal). Second, detection condition 5 of Theorem 4.18(a) (that is, "$\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1} = 1$") differs from detection condition 4 in Theorem 4.18(b) syntactically. We say that these two conditions are syntactically different because they are actually equivalent to each other when $k_{i_1} = 1$. It is because of the following reason

$$
\begin{aligned}
& \vec{t} \in FP(S) \text{ such that } x_{l_1} = 1 \\
\equiv\ & \vec{t} \in FP(S) \text{ such that } p_{i_1} = x_1^{i_1} = 0 \text{ and } x_{l_1} = 1 \\
\equiv\ & \vec{t} \in FP(S) \text{ such that } p_{i_1,\bar{1}} = \bar{x}_1^{i_1} = 1 \text{ and } x_{l_1} = 1 \\
\equiv\ & \vec{t} \in NFP_{i_1,\bar{1}}(S) \text{ such that } x_{l_1} = 1 \qquad \text{(Please be noted that } j_1 = 1 \text{ when } k_{i_1} = 1\text{)}
\end{aligned}
$$

Hence, without loss of generality, we can still use the seven detection conditions in Theorem 4.18(a) to represent the detection conditions of double-fault expression (18) in Table 1 for both situations in Theorem 4.18(a) and (b), bearing in mind the non-existence of the term $p_{i_1,\hat{j}_1}$ and the equivalence between "$\vec{t} \in FP(S)$ such that $x_{l_1} = 1$" and "$\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1} = 1$" when $k_{i_1} = 1$. Similarly, we can use the seven detection conditions in Theorem 4.18(a) to represent the detection conditions of double-fault expression (18) in Table 1 for situations in Theorem 4.18(a), (b), (c) and (d), bearing in mind the non-existence of the term $p_{i_2,\hat{j}_2}$ and the equivalence between "$\vec{t} \in FP(S)$ such that $x_{l_2} = 1$" and "$\vec{t} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$" when $k_{i_2} = 1$.

Theorem 4.19 *(LRF with LRF - Case 2)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two literals $x_{j_1}^{i_1}$ and $x_{j_2}^{i_1}$ in the i_1-th term, p_{i_1}, in S are replaced by x_{l_1} and x_{l_2}, respectively, where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 \leq k_{i_1}$, k_{i_1} is the number of literals of p_{i_1} and x_{l_1} and x_{l_2} are two different missing literals of p_{i_1} from different Boolean variables.

(a) When $k_{i_1} > 2$, the resulting expression denoted as $I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$ is equivalent to double-fault expression (19) in Table 1. Then, $S \not\equiv I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$ if

and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ *such that* $x_{l_1}x_{l_2} = 0$,
2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ *such that* $x_{l_1}x_{l_2} = 1$,
3. $\vec{t} \in NFP_{i_1,\bar{j}_2}(S)$ *such that* $x_{l_1}x_{l_2} = 1$, *or*
4. $\vec{t} \in FP(S)$ *such that* $p_{i_1,\bar{j}_1,\bar{j}_2} = 1$ *and* $x_{l_1}x_{l_2} = 1$.

(b) *When* $k_{i_1} = 2$, *the resulting expression denoted as* $I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1}x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2}x_{l_2})}$ *is equivalent to that given by double-fault expression (19) in Table 1 without* $p_{i_1,\hat{j}_1,\hat{j}_2}$ *because* p_{i_1} *contains just two literals. Then,* $S \not\equiv I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1}x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2}x_{l_2})}$ *if and only if there is a test case* $\vec{t}$ *that satisfies any of the following conditions:*

1. $\vec{t} \in UTP_{i_1}(S)$ *such that* $x_{l_1}x_{l_2} = 0$ *or*
2. $\vec{t} \in FP(S)$ *such that* $x_{l_1}x_{l_2} = 1$.

Proof : (a) First, we observe that $S \oplus I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1}x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2}x_{l_2})}$

$\equiv (p_{i_1} \oplus p_{i_1,\hat{j}_1,\hat{j}_2}x_{l_1}x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1}\overline{p_{i_1,\hat{j}_1,\hat{j}_2}x_{l_1}x_{l_2}} + \bar{p}_{i_1}p_{i_1,\hat{j}_1,\hat{j}_2}x_{l_1}x_{l_2}\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv (p_{i_1}\overline{x_{l_1}x_{l_2}} + \bar{p}_{i_1}p_{i_1,\hat{j}_1,\hat{j}_2}x_{l_1}x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

(By making use of $ABC(\overline{ADE}) \equiv ABC(\overline{DE})$)

$\equiv (p_{i_1}\overline{x_{l_1}x_{l_2}} + \bar{p}_{i_1}p_{i_1,\bar{j}_1}x_{l_1}x_{l_2} + \bar{p}_{i_1}p_{i_1,\bar{j}_2}x_{l_1}x_{l_2} + \bar{p}_{i_1}p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_1}x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

(By rewriting $(\overline{ABC})A$ as $(\overline{ABC})A(\overline{B})C + (\overline{ABC})AB(\overline{C}) + (\overline{ABC})A(\overline{B})(\overline{C})$

because they are equivalent)

$\equiv p_{i_1}\overline{x_{l_1}x_{l_2}}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}x_{l_1}x_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$+p_{i_1,\bar{j}_2}x_{l_1}x_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_1}x_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\overline{x_{l_1}x_{l_2}}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}x_{l_1}x_{l_2}\overline{S} + p_{i_1,\bar{j}_2}x_{l_1}x_{l_2}\overline{S} + p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_1}x_{l_2}\overline{S}$

Now,	$S(\vec{t}) \neq I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}(\vec{t}) = 1$
if and only if	$p_{i_1}\overline{x_{l_1}x_{l_2}}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1,\bar{j}_1}x_{l_1}x_{l_2}\overline{S} + p_{i_1,\bar{j}_2}x_{l_1}x_{l_2}\overline{S} + p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_1}x_{l_2}\overline{S}$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1}x_{l_2} = 0$,
2. $\vec{t} \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1}x_{l_2} = 1$,
3. $\vec{t} \in NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1}x_{l_2} = 1$, or
4. $\vec{t} \in FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} = 1$ and $x_{l_1}x_{l_2} = 1$.

Hence, the result follows.

(b) The proof of this part is similar to (a) above except that the term $p_{i_1,\hat{j}_1,\hat{j}_2}$ does not appear in the proof. We proceed the proof as follows.

First, we observe that $S \oplus I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}$

$\equiv (p_{i_1} \oplus x_{l_1}x_{l_2})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1}\overline{x_{l_1}x_{l_2}} + \bar{p}_{i_1}x_{l_1}x_{l_2}\right)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\overline{x_{l_1}x_{l_2}}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + \bar{p}_{i_1}x_{l_1}x_{l_2}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\overline{x_{l_1}x_{l_2}}\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m + x_{l_1}x_{l_2}\overline{S}$

Now, $S(\vec{t}) \neq I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{LRF(p_{i_1} \to p_{i_1,\hat{j}_1} x_{l_1}) \bowtie LRF(p_{i_1} \to p_{i_1,\hat{j}_2} x_{l_2})}(\vec{t}) = 1$

if and only if $p_{i_1} \overline{x_{l_1} x_{l_2}} \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+1} \cdots \bar{p}_m + x_{l_1} x_{l_2} \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $x_{l_1} x_{l_2} = 0$ or

2. $\vec{t} \in FP(S)$ such that $x_{l_1} x_{l_2} = 1$.

Hence, the result follows. □

It should be noted that detection conditions 2, 3 and 4 of Theorem 4.19(a) are just syntactically different from the detection condition 2 of Theorem 4.19(b). In fact, they are actually equivalent to each other when $k_{i_1} = 2$ (that is p_{i_1} contains just two literals). The reason is similar to that in the paragraph after Theorem 4.13. Hence, without loss of generality, we can still use the four detection conditions in Theorem 4.19(a) to represent the detection conditions of double-fault expression (19) in Table 1 for $k_{i_1} \geq 2$, bearing in mind that the detection conditions 2, 3 and 4 degenerate to "$\vec{t} \in FP(S)$ such that $x_{l_1} x_{l_2} = 1$" when $k_{i_1} = 2$.

In summary, for ease of reference, we list all double fault classes and their corresponding fault detection conditions in Table 2. Let us consider the third row of Table 2, which presents the detection conditions of two double-fault expressions of LNF ⋈ LIF. For double-fault expression (5) (please refer to Table 1 for the actual double-fault expression), the detection condition shows that "any point in $TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ such that $p_{i_2} x_{l_2} = 0$", "any point in $TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big)$ such that $p_{i_1,\bar{j}_1} + x_{l_2} = 0$", or "any point in $NFP_{i_1,\bar{j}_1}(S)$" can distinguish between S and the double-fault expression (5).

Table 2: Double fault, double-fault expression and detection condition ($S = p_1 + \ldots + p_m$)

Fault Class	(Expression No.): Detection Condition
LNF ⋈ LNF	(1): (C1) any point in $\left(TP_{i_1}(S) \setminus \left(\bigcup_{i=1, i \neq i_1, i_2}^{m} TP_i(S)\right)\right)$ such that $p_{i_2,\bar{j}_2}=0$ or (C2) any point in $\left(TP_{i_2}(S) \setminus \left(\bigcup_{i=1, i \neq i_1, i_2}^{m} TP_i(S)\right)\right)$ such that $p_{i_1,\bar{j}_1}=0$ or (C3) any point in $NFP_{i_1,\bar{j}_1}(S)$ or (C4) any point in $NFP_{i_2,\bar{j}_2}(S)$
	(2): (C1) any point in $UTP_{i_1}(S)$ or (C2) any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} = 1$
LNF ⋈ LOF	(3): (C1) any point in $UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} = 0$ or (C2) any point in $NFP_{i_1,\bar{j}_1}(S)$ or (C3) any point in $NFP_{i_2,\bar{j}_2}(S)$
	(4): (C1) any point in $UTP_{i_1}(S)$ or (C2) any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\hat{j}_2} = 1$
LNF ⋈ LIF	(5): (C1) any point in $\left(TP_{i_1}(S) \setminus \left(\bigcup_{i=1, i \neq i_1, i_2}^{m} TP_i(S)\right)\right)$ such that $p_{i_2}x_{l_2} = 0$ or (C2) any point in $\left(TP_{i_2}(S) \setminus \left(\bigcup_{i=1, i \neq i_1, i_2}^{m} TP_i(S)\right)\right)$ such that $p_{i_1,\bar{j}_1} + x_{l_2} = 0$ or (C3) any point in $NFP_{i_1,\bar{j}_1}(S)$
	(6): (C1) any point in $UTP_{i_1}(S)$ or (C2) any point in $NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$
LNF ⋈ LRF	(7): (C1) any point in $\left(TP_{i_1}(S) \setminus \left(\bigcup_{i=1, i \neq i_1, i_2}^{m} TP_i(S)\right)\right)$ such that $p_{i_2,\hat{j}_2}x_{l_2} = 0$ or (C2) any point in $\left(TP_{i_2}(S) \setminus \left(\bigcup_{i=1, i \neq i_1, i_2}^{m} TP_i(S)\right)\right)$ such that $p_{i_1,\bar{j}_1} + x_{l_2} = 0$ or (C3) any point in $NFP_{i_1,\bar{j}_1}(S)$ or (C4) any point in $NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$
	(8): (C1) any point in $UTP_{i_1}(S)$ or (C2) any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\hat{j}_2}x_{l_2} = 1$
LOF ⋈ LOF	(9): (C1) any point in $NFP_{i_1,\bar{j}_1}(S)$ or (C2) any point in $NFP_{i_2,\bar{j}_2}(S)$
	(10): (C1) any point in $FP(S)$ such that $p_{i_1,\hat{j}_1,\hat{j}_2} = 1$

Table 2 (cont'd) Double fault, double-fault expression and detection condition ($S = p_1 + \ldots + p_m$)

Fault Class	(Expression No.): Detection Condition
LOF ⋈ LIF	(11): (C1) any point in $UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$ or (C2) any point in $NFP_{i_1,\bar{j}_1}(S)$
LOF ⋈ LRF	(12): (C1) any point in $UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$ or (C2) any point in $NFP_{i_1,\bar{j}_1}(S)$ or (C3) any point in $NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$
	(13): (C1) any point in $UTP_{i_1}(S)$ such that $x_{l_2} = 0$ or (C2) any point in $NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$ or (C3) any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_2} = 1$ or (C4) any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} x_{l_2} = 1$
LIF ⋈ LIF	(14): (C1) any point in $UTP_{i_1}(S)$ such that $x_{l_1} = 0$ or (C2) any point in $UTP_{i_2}(S)$ such that $x_{l_2} = 0$ or (C3) any point in $\Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S)\big)\Big)$ such that $x_{l_1} + x_{l_2} = 0$
	(15): (C1) any point in $UTP_{i_1}(S)$ such that $x_{l_1} x_{l_2} = 0$
LIF ⋈ LRF	(16): (C1) any point in $UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$ or (C2) any point in $UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$ or (C3) any point in $UTP_{i_2}(S)$ such that $x_{l_2} = 0$ or (C4) any point in $NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$ or (C5) any point in $\Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S)\big)\Big)$ such that $x_{l_1} + x_{l_2} = 0$
	(17): (C1) any point in $UTP_{i_1}(S)$ such that $x_{l_1} x_{l_2} = 0$ or (C2) any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1} x_{l_2} = 1$
LRF ⋈ LRF	(18): (C1) any point in $UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$ or (C2) any point in $UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$ or (C3) any point in $UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$ or (C4) any point in $UTP_{i_2}(S)$ such that $x_{l_1} + x_{l_2} = 0$ or (C5) any point in $NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1} = 1$ or (C6) any point in $NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$ or (C7) any point in $\Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S)\big)\Big)$ such that $x_{l_1} + x_{l_2} = 0$
	(19): (C1) any point in $UTP_{i_1}(S)$ such that $x_{l_1} x_{l_2} = 0$ or (C2) any point in $NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1} x_{l_2} = 1$ or (C3) any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1} x_{l_2} = 1$ or (C4) any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} = 1$ and $x_{l_1} x_{l_2} = 1$

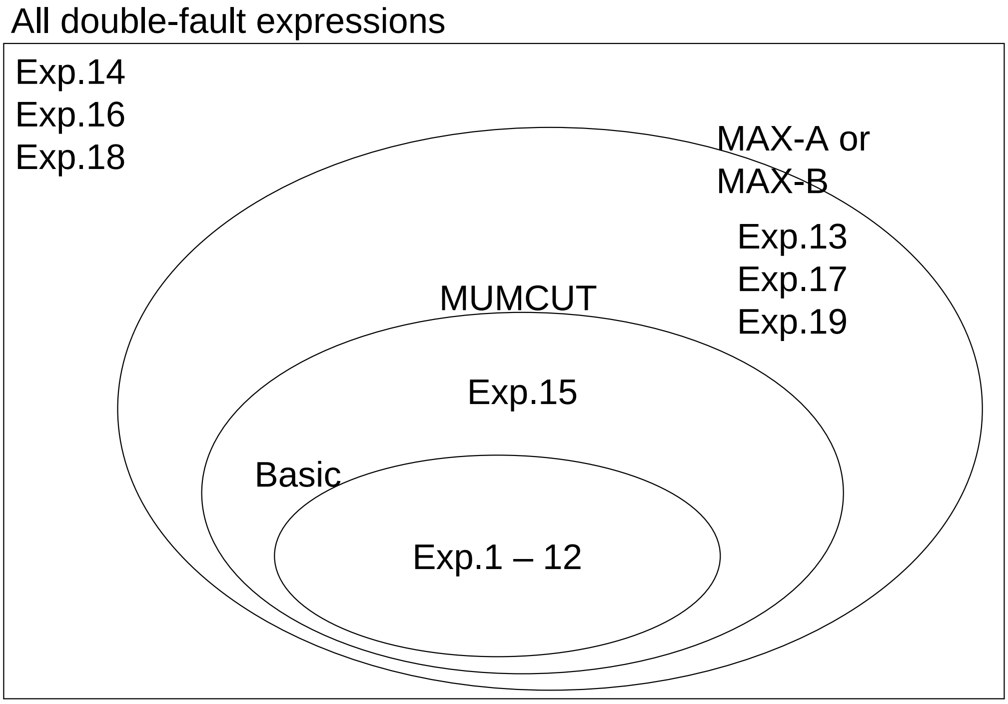

Figure 1: Double-fault expressions detected by each strategy

5 Fault Detection Capability of Existing Strategies

In this section, we analyse the fault detecting capability of the strategies as mentioned in Section 2.3, namely, the BASIC, MUMCUT, MAX-A and MAX-B strategies, with respect to all 19 double-fault expressions related to double literal faults. As a reminder, please note that the MAX-B strategy subsumes the MAX-A strategy, which in turn subsumes the MUMCUT strategy, which in turn subsumes the BASIC strategy.

Figure 1 indicates the double-fault expressions studied in this report that can be detected by each strategy and those that cannot be detected.[1] Hence, it summarizes our results in this section. The discussion is outlined as follows

[1] In an earlier version of this technical report, we cannot prove that the MAX-A strategy, and hence the MAX-B strategy, can detect double-fault expressions (13) and (19). Nor can we find counterexamples that the MAX-B strategy, and hence the MAX-A strategy, cannot detect these two double-fault expressions. However, in this revised version, we are able to prove that the MAX-A strategy (and hence, the MAX-B strategy) can detect double-fault expression (13) in Theorem 5.7, (19) in Theorem 5.9. Theorem 5.7 relies on Lemma 5.1 which is quite tricky to prove, and Theorem 5.9 on Lemma 5.2. Please see the proofs of these two lemmas in detail.

1. In Section 5.1, we show that the BASIC strategy can detect double-fault expressions (1) to (12). Furthermore, we illustrate using example that the BASIC strategy cannot detect double-fault expression (15).

2. In Section 5.2, we show that the MUMCUT strategy can further detect double-fault expression (15) in addition to double-fault expressions (1)–(12), As a result, the MUMCUT strategy can detect 13 double-fault expressions. Furthermore, we illustrate using examples that the MUMCUT strategy cannot detect double-fault expressions (13), (17) and (19).

3. In Section 5.3, we show that the MAX-A strategy can further detect double-fault expressions (13), (17) and (19) in addition to detecting all double-fault expressions detected by the MUMCUT strategy. As a result, the MAX-A strategy can detect 16 double-fault expressions.

4. In Section 5.4, we illustrate using examples that the MAX-B strategy cannot detect double-fault expressions (14), (16) and (18). Hence, the MAX-B strategy can detect the same 16 double-fault expressions that can be detected by the MAX-A strategy.

5.1 The BASIC Strategy

In this section, we prove in Theorems 5.1 to 5.2 that, for double-fault expressions (1) to (12) in Table 1, the test set selected by the BASIC strategy satisfies their detection conditions in Table 2.

Table 3 summarizes the double-fault expressions and the corresponding detection conditions satisfied by the BASIC strategy. As a result, any strategy that subsumes the BASIC strategy can detect these 12 double-fault expressions.

Theorem 5.1 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Suppose that T is the set of near false points formed by selecting a near false point from every $NFP_{i,\bar{j}}(S)$. Then T satisfies the following conditions:*

Table 3: Conclusions in Theorems 5.1 and 5.2 satisfying detection conditions in Table 2

Double-fault Expression	Detection Condition	Conclusion in	
		Theorem 5.1	Theorem 5.2
(1)	(C3)	(1)	–
	(C4)	(1)	–
(2)	(C1)	–	(1)
(3)	(C2)	(1)	–
	(C3)	(1)	–
(4)	(C1)	–	(1)
	(C2)	(2)	–
(5)	(C1)	–	(2)
	(C3)	(1)	–
(6)	(C1)	–	(1)
(7)	(C3)	(1)	–
(8)	(C1)	–	(1)
(9)	(C1)	(1)	–
	(C2)	(1)	–
(10)	(C1)	(3)	–
(11)	(C2)	(1)	–
(12)	(C2)	(1)	–

1. *There exists $\vec{t} \in T$ such that $\vec{t} \in NFP_{i,\bar{j}}(S)$ for any i and j where $1 \le i \le m$ and $1 \le j \le k_i$.*

2. *There exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ such that $p_{i,\bar{j}_1,\hat{j}_2} = 1$ where $p_{i,\bar{j}_1,\hat{j}_2}$ can be obtained from p_i by negating $x^i_{j_1}$ and omitting $x^i_{j_2}$, $1 \le i \le m$ and $1 \le j_1 < j_2 \le k_i$.*

3. *There exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ such that $p_{i,\hat{j}_1,\hat{j}_2} = 1$ where $p_{i,\hat{j}_1,\hat{j}_2}$ can be obtained from p_i by omitting $x^i_{j_1}$ and $x^i_{j_2}$, $1 \le i \le m$, $1 \le j_1 < j_2 \le k_i$ and $k_i > 2$.*

Proof : Since S is in IDNF, $NFP_{i,\bar{j}}(S) \ne \emptyset$ for every i and j.

1. By definition of T, there exists $\vec{t} \in T$ such that $\vec{t} \in NFP_{i,\bar{j}}(S)$.

2. By definition of T, there exists $\vec{t} \in T$ such that $\vec{t} \in NFP_{i,\bar{j}_1}(S) \subset FP(S)$. Therefore, all literals of p_i (that is, $x^i_1, \cdots, x^i_{k_{i_1}}$) evaluate to 1 except $x^i_{j_1}$ which evaluates to 0 on $\vec{t}$. Therefore, $p_{i,\bar{j}_1,\hat{j}_2} = x^i_1 \cdots \bar{x}^i_{j_1} \cdots x^i_{j_2-1} x^i_{j_2+1} \cdots x^i_{k_{i_1}} = 1$.

3. By definition of T, there exists $\vec{t} \in T$ such that $\vec{t} \in NFP_{i,\bar{j}_1}(S) \subset FP(S)$. Therefore, all literals of p_i (that is, $x^i_1, \cdots, x^i_{k_i}$) evaluate to 1 except $x^i_{j_1}$ which evaluates to 0 on $\vec{t}$. Therefore, $p_{i,\hat{j}_1,\hat{j}_2} = x^i_1 \cdots x^i_{j_1-1} x^i_{j_1+1} \cdots x^i_{j_2-1} x^i_{j_2+1} \cdots x^i_{k_{i_1}} = 1$.

Hence, the result follows. □

Theorem 5.2 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Suppose that T is the set of unique true points formed by selecting a unique true point from every $UTP_i(S)$. Then T satisfies the following conditions*

1. *There exists $\vec{t} \in T$ such that $\vec{t} \in UTP_i(S)$ for any i where $1 \le i \le m$.*

2. *There exists $\vec{t} \in T$ such that $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{i=1, i \ne i_1, i_2}^{m} TP_i(S) \big)$ such that $p_{i_2} x_{l_2} = 0$ where x_{l_2} is a missing literal of p_{i_2} and $1 \le i_1 < i_2 \le m$.*

Proof : Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for every i.

1. By definition of T, there exists $\vec{t} \in T$ such that $\vec{t} \in UTP_i(S)$.
2. By definition of T, there exists $\vec{t} \in T$ such that $\vec{t} \in UTP_{i_1}(S) = \Big(TP_{i_1}(S) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S))\Big) \subseteq \Big(TP_{i_1}(S) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S))\Big)$ because $(\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)) \supseteq (\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S))$. Therefore, p_{i_1} evaluates to 1 and all other terms of S evaluate to 0 on $\vec{t}$. Thus, $p_{i_2}x_{l_2} = 0 \cdot x_{l_2} = 0$.

Hence, the result follows. □

Theorem 5.3 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. The BASIC meaningful impact strategy can detect double-fault expressions (1)–(12) in Table 1.*

Proof : Let T be a test set generated by the BASIC strategy. Therefore, T contains a test case from every $UTP_i(S)$ and a test case from every $NFP_{i,\bar{j}}(S)$. Hence, by Theorems 5.1 and 5.2, T satisfies the conclusions in these theorems. Now, Table 3 indicates that these conclusions satisfy the detection conditions of double-fault expressions (1) to (12). As a result, T can detect all these double fault expressions. Hence, the result follows. □

Theorem 5.4 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Any test case selection strategy that subsumes the BASIC meaningful impact strategy can detect double-fault expressions (1)–(12) in Table 1.*

Proof : By Theorem 5.3 and the definition of subsumption of test case selection strategy. □

In the following, we use an example to illustrate that the BASIC strategy cannot detect double-fault expression (15).

Table 4: All unique true points and near false points of $S = ab + cde$

i	1	2
$UTP_i(S)$	11000, 11001, 11010, 11011, 11100, 11101, 11110	00111, 01111, 10111
$NFP_{i,\bar{1}}(S)$	01000, 01001, 01010, 01011, 01100, 01101, 01110	00011, 01011, 10011
$NFP_{i,\bar{2}}(S)$	10000, 10001, 10010, 10011, 10100, 10101, 10110	00101, 01101, 10101
$NFP_{i,\bar{3}}(S)$	–	00110, 01110, 10110

Example 5.1 Let $S = ab + cde$. Table 4 lists the sets $UTP_i(S)$ of all unique true points for every possible i and the sets $NFP_{i,\bar{j}}(S)$ of all near false points for every possible i and j pair. Suppose the literals c and d are inserted into the first term of S, the resulting double-fault expression is equivalent to $I = abcd + cde$, which is a special instance of double-fault expression (15). Note that S and I are not equivalent because S and I evaluate to 1 and 0 on 11000, respectively.

Let T be the set {11110, 00111, 01001, 10001, 00011, 00101, 00110}. Test cases in T are underlined in Table 4 for ease of reference. T satisfies the BASIC strategy. However, T cannot distinguish between S and I because S and I agree on all points in T. ◊

5.2 The MUMCUT Strategy

Since the MUMCUT strategy subsumes the BASIC strategy, by Theorem 5.4, it also detects the 12 double-fault expressions that the BASIC strategy detects. In the following, we show that the MUMCUT strategy can detect double-fault expression (15). More precisely, the test set generated by the MUMCUT strategy satisfies the detection condition of double-fault expression (15). As a result, the MUMCUT strategy detects a total of 13 double-fault expressions.

Theorem 5.5 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal* *form. Suppose that two different missing literals x_{l_1} and x_{l_2} of the term p_{i_1} is inserted into the term, the resulting expression denoted as I is equivalent to double-fault expression (15) in Table 1. The MUTP strategy can detect double-fault expression (15) provided that $S \not\equiv I$.*

Proof : Since $S \not\equiv I$, by Theorem 4.15, there is a test case $\vec{t_0} \in UTP_{i_1}(S)$ such that $x_{l_1}x_{l_2} = 0$. Hence, we have the following two cases

1. $\vec{t_0}$ is such that $x_{l_1} = 0$. Then, the set $X = \{\vec{t} \in UTP_{i_1}(S) : x_{l_1} = 0\}$ is non-empty. As a result, the MUTP strategy can select a point from $UTP_{i_1}(S)$ such that $x_{l_1} = 0$.
2. $\vec{t_0}$ is such that $x_{l_2} = 0$. The proof is similar to 1 above.

Hence, the result follows. □

Theorem 5.6 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Any test case selection strategy that subsumes the MUMCUT meaningful impact strategy can detect double-fault expressions (1)–(12) and (15) in Table 1.*

Proof : By Theorems 5.4 and 5.5 and the definition of subsumption. □

We now illustrate that the MUMCUT strategy cannot detect double-fault expressions (13), (17) and (19) in Examples 5.2, 5.3 and 5.4, respectively.

Example 5.2 Let $S = abcd + abc\bar{e} + \bar{a}be + a\bar{b}e$. Table 5 lists the sets $UTP_i(S)$ of all unique true points for every possible i and the sets $NFP_{i,\bar{j}}(S)$ of all near false points for every possible i and j pair. Suppose that the literal a of the first term $abcd$ is omitted from the term and the literal b of $abcd$ is replaced by the literal e, the resulting double-fault expression is equivalent to $I = cde + abc\bar{e} + \bar{a}be + a\bar{b}e$, which is a special instance of double-fault expression (13). Note that S and I are not equivalent because S and I evaluate to 0 and 1 on 00111, respectively.

Table 5: All unique true points and near false points of S where $S = abcd + abc\bar{e} + \bar{a}be + a\bar{b}e$

i	1	2	3	4
$UTP_i(S)$	11111	11100	01001, 01011, 01101, 01111	10001, 10011, 10101, 10111
$NFP_{i,\bar{1}}(S)$	01110	01100, 01110	11001, 11011, 11101	00001, 00011, 00101, 00111
$NFP_{i,\bar{2}}(S)$	10110	10100, 10110	00001, 00011, 00101, 00111	11001, 11011, 11101
$NFP_{i,\bar{3}}(S)$	11010, 11011	11000, 11010	01000, 01010, 01100, 01110	10000, 10010, 10100, 10110
$NFP_{i,\bar{4}}(S)$	11101	11101	–	–

Let T be the set {11111, 01110, 10110, 11010, 11011, 11101, 11100, 01100, 10100, 11000, 01011, 01101, 00011, 00101, 01010, 10011, 10101, 10010}. Test cases in T are underlined in Table 5 for ease of reference. The set T satisfies the MUMCUT strategy. However, T cannot be used to detect I because S and I agree on all points in T. ◊

Example 5.3 Let $S = abc + bc\bar{d} + bc\bar{e} + \bar{b}e\bar{f} + \bar{c}e\bar{f} + bcf$. Table 6 lists the sets $UTP_i(S)$ of all unique true points for every possible i and the sets $NFP_{i,\bar{j}}(S)$ of all near false points for every possible i and j pair. Suppose that the literal d is inserted into the first term abc and the literal c in abc is replaced by the literal e, the resulting double-fault expression, denoted as I, is equivalent to $abde + bc\bar{d} + bc\bar{e} + \bar{b}e\bar{f} + \bar{c}e\bar{f} + bcf$, which is a special instance of double-fault expression (17). Note that S and I are not equivalent because S and I evaluate to 0 and 1 on 110111, respectively.

Let T be the set {111110, 011110, 101011, 101100, 110011, 110100, 011010, 001000, 010000, 011100, 001100, 101001, 010100, 110001, 001110, 101010, 101000, 001111, 010110, 110010, 110000, 010111, 011111}. Test cases in T are underlined in Table 6 for ease of reference. The set T satisfies the MUMCUT strategy. However, T cannot be used to detect I because S and I agree on all points in T. ◊

Example 5.4 Let $S = abc + bc\bar{d} + bc\bar{e} + \bar{b}e\bar{f} + \bar{c}e\bar{f} + bcf$, which is the same expression as in Example 5.3. Suppose that the literals b and c in the first term abc are replaced by the literals d and

Table 6: All unique true points and near false points of $S(=abc+bc\bar{d}+bc\bar{e}+\bar{b}e\bar{f}+\bar{c}e\bar{f}+bcf)$

i	1	2	3	4	5	6
$UTP_i(S)$	111110	011010	011100	001010, 001110, 101010, 101110	010010, 010110, 110010, 110110	011111
$NFP_{i,\bar{1}}(S)$	011110	001000, 001001, 001011, 101000, 101001, 101011	001000, 001001, 001100, 001101, 101000, 101001, 101100, 101101	011110	011110	001001, 001011, 001101, 001111, 101001, 101011, 101101, 101111
$NFP_{i,\bar{2}}(S)$	101000 101001, 101011, 101100, 101101, 101111,	010000, 010001, 010011, 110000, 110001, 110011	010000, 010001, 010100, 010101, 110000, 110001, 110100, 110101	000000, 000100, 001000, 001100, 100000, 100100, 101000, 101100	000000, 000100, 010000, 010100, 100000, 100100, 110000, 110100	010001, 010011, 010101, 010111, 110001, 110011, 110101, 110111
$NFP_{i,\bar{3}}(S)$	110000, 110001, 110011, 110100, 110101, 110111	011110	011110	000011, 000111, 001011, 001111, 100011, 100111, 101011, 101111	000011, 000111, 010011, 010111, 100011, 100111, 110011, 110111	011110

e, respectively, the resulting double-fault expression is equivalent to $I = ade + bc\bar{d} + bc\bar{e} + \bar{b}e\bar{f} + \bar{c}e\bar{f} + bcf$, which is a special instance of double-fault expression (19). Note that S and I are not equivalent because S and I evaluate to 0 and 1 on 110111, respectively.

Let T be the same test set as in Example 5.3 that satisfies the MUMCUT strategy. However, T cannot be used to detect I because S and I agree on all points in T. ◊

5.3 The MAX-A Strategy

In this section, we prove that the MAX-A strategy can further detect double-fault expressions (13), (17) and (19) in addition to those detected by the MUMCUT strategy. More precisely, Theorems 5.7, 5.8 and 5.9 show that test sets generated by the MAX-A strategy satisfy the detection conditions of double-fault expressions (13), (17) and (19), respectively.

We need the following lemma before proving Theorem 5.7.

Lemma 5.1 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the j_1-th literal, $x^{i_1}_{j_1}$, of the i_1-th term, p_{i_1}, in S is omitted from the term and the j_2-th literal $x^{i_1}_{j_2}$ in p_{i_1} is replaced by x_{l_2} where $1 \le i_1 \le m$, $1 \le j_1 < j_2 \le k_{i_1}$, and x_{l_2} is a missing literal of p_{i_1}, the resulting implementation denoted by I is equivalent to that given by double-fault expression (13) in Table 1 and its detection conditions are given by (C1) to (C4) of double-fault expression (13) in Table 2. If detection conditions (C1), (C2) and (C3) cannot be satisfied, any test case that satisfies detection condition (C4) is a near false point of S provided that $S \not\equiv I$.*

Proof : Since $S \not\equiv I$, there exists a test case that can satisfy any one of the following detection conditions of I as given in Table 2 for double-fault expression (13):

(C1) any point in $UTP_{i_1}(S)$ such that $x_{l_2} = 0$, (C2) any point in $NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$,

(C3) any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or

(C4) any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_2} = 1$.

Based on the assumption of the lemma, (C1) to (C3) cannot be satisfied. The test case satisfies (C4). In other words, the set of points that satisfy (C4) is non-empty.

Let $\vec{t}$ be a test case that satisfies (C4). We now prove that $\vec{t}$ is a near false point of S.

We observe that $p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_2}(\vec{t}) = x_1^{i_1}\cdots\bar{x}_{j_1}^{i_1}\cdots\bar{x}_{j_2}^{i_1}\cdots x_{k_{i_1}}^{i_1}x_{l_2}(\vec{t}) = 1$. Now, let $\vec{t}_1$ be such that $\vec{t}_1$ and $\vec{t}$ only differ at the truth value of the literal $x_{j_2}^{i_1}$. Then, $p_{i_1,\bar{j}_1}x_{l_2}(\vec{t}_1) = x_1^{i_1}\cdots\bar{x}_{j_1}^{i_1}\cdots x_{j_2}^{i_1}\cdots x_{k_{i_1}}^{i_1}x_{l_2}(\vec{t}_1) = 1$.

Please note that $\vec{t}_1$ is a true point of S. Suppose on the contrary that $S(\vec{t}_1) = 0$. In that case, $p_{i_1,\bar{j}_1}(\vec{t}_1) = 1$ and $x_{l_2}(\vec{t}_1) = 1$ because $p_{i_1,\bar{j}_1}x_{l_2}(\vec{t}_1) = 1$. As a result, $\vec{t}_1 \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$. We found a test case $\vec{t}_1$ satisfying (C2). This violates the assumption that (C2) cannot be satisfied.

Since $p_{i_1,\bar{j}_1}(\vec{t}_1) = 1$, $p_{i_1}(\vec{t}_1) = 0$. In other words, $\vec{t}_1$ is a true point of S but not a true point for p_{i_1}. Hence, there exists $i_3 (\neq i_1)$ such that $\vec{t}_1$ is a true point of p_{i_3} (that is, $p_{i_3}(\vec{t}_1) = 1$).

We now prove that $x_{j_2}^{i_1}$ exists in p_{i_3}. First, if both $x_{j_2}^{i_1}$ and $\bar{x}_{j_2}^{i_1}$ do not appear in p_{i_3}, then $p_{i_3}(\vec{t}) = p_{i_3}(\vec{t}_1) = 1$ because $\vec{t}$ and $\vec{t}_1$ only differ at $x_{j_2}^{i_1}$. Hence, $\vec{t}$ is a true point of S. This contradicts to the fact that $\vec{t}$ is a false point of S. Second, if $\bar{x}_{j_2}^{i_1}$ appears in p_{i_3}, then $p_{i_3}(\vec{t}_1) = \cdots\bar{x}_{j_2}^{i_1}\cdots = 0$ because $p_{i_1,\bar{j}_1}(\vec{t}_1) = 1$ implies $x_{j_2}^{i_1}(\vec{t}_1) = 1$. This contradicts to the fact that $p_{i_3}(\vec{t}_1) = 1$. Therefore $x_{j_2}^{i_1}$ must appear in p_{i_3}.

Since $x_{j_2}^{i_1}$ is a literal of p_{i_3}, there exists j_3 such that $x_{j_2}^{i_1}$ is the j_3-th literal of p_{i_3} (that is, $x_{j_3}^{i_3} = x_{j_2}^{i_1}$). Now, since $\vec{t}_1$ and $\vec{t}$ only differ at $x_{j_3}^{i_3}$ ($= x_{j_2}^{i_1}$) and $p_{i_3}(\vec{t}_1) = 1$, $p_{i_3,\bar{j}_3}(\vec{t}) = x_1^{i_3}\cdots\bar{x}_{j_3}^{i_3}\cdots x_{k_{i_3}}^{i_3}(\vec{t}) = 1$. In summary, $\vec{t}$ is a false point of S and $p_{i_3,\bar{j}_3}(\vec{t}) = 1$. Therefore, $\vec{t} \in NFP_{i_3,\bar{j}_3}(S)$. Hence, the result follows. □

We prove Theorem 5.7 which states that the MAX-A strategy detects double-fault expression (13).

Theorem 5.7 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that the j_1-th literal, $x^{i_1}_{j_1}$, of the i_1-th term, p_{i_1}, in S is omitted from the term and the j_2-th literal, $x^{i_1}_{j_2}$, in p_{i_1} is replaced by x_{l_2} where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 \leq k_{i_1}$, and x_{l_2} is a missing literal of p_{i_1}, the resulting implementation denoted by I is equivalent to that given by double-fault expression (13) in Table 1 and its detection conditions are given by (C1) to (C4) of double-fault expression (13) in Table 2. The MAX-A strategy can guarantee to detect double-fault expression (13) provided that $S \not\equiv I$.*

Proof : Detection conditions (C1) to (C4) of double-fault expression (13) are listed below

(C1) any point in $UTP_{i_1}(S)$ such that $x_{l_2} = 0$,

(C2) any point in $NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$,

(C3) any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or

(C4) any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_2} = 1$.

We proceed our proof using the following two cases

Case 1. Any one of the conditions (C1) to (C3) can be satisfied. Since $S \not\equiv I$, there is a test case $\vec{t}$ that satsifies (C1), (C2) or (C3). We then have the following three subcases

(a) If $\vec{t}$ satisfies (C1), $\vec{t}$ must be a unique true point.

(b) If $\vec{t}$ satisfies (C2), $\vec{t}$ must be a near false point.

(c) If $\vec{t}$ satisfies (C3), $\vec{t}$ must be a near false point.

Hence, the MAX-A strategy can select $\vec{t}$.

Case 2. Conditions (C1) to (C3) cannot be satisfied. Since $S \not\equiv I$, there exists a test case $\vec{t}$ that satisfies (C4). By Lemma 5.1, $\vec{t}$ is a near false point of S. Hence, the MAX-A strategy can select $\vec{t}$ because it selects all unique true points and all near false points of S.

Hence, The result follows. □

We prove Theorem 5.8 which states that the MAX-A strategy detects double-fault expression (17).

Theorem 5.8 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that the literal x_{l_1} is inserted in the i_1-th term, p_{i_1}, in S and the j_2-th literal, $x^{i_1}_{j_2}$, in p_{i_1} is replaced by x_{l_2} where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 \leq k_{i_1}$, and x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}, the resulting implementation denoted by I is equivalent to that given by double-fault expression (17) in Table 1 and its detection conditions are given by (C1) to (C3) of double-fault expression (17) in Table 2. The MAX-A strategy can guarantee to detect double-fault expression (17) provided that $S \not\equiv I$.*

Proof : As a reminder, the detection conditions (C1) to (C3) of double-fault expression (17) are listed below

(C1) any point in $UTP_{i_1}(S)$ such that x_{l_1} = 0,

(C2) any point in $UTP_{i_1}(S)$ such that x_{l_2} = 0, or

(C3) any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1}x_{l_2}$ = 1.

We proceed the proof as follows. Since $S \not\equiv I$, there exists a test case $\vec{t}$ such that it satisfy any one of (C1) to (C3). We have the following three cases

1. If $\vec{t}$ satisfies (C1), it is a unique true point.
2. If $\vec{t}$ satisfies (C2), it is a unique true point.
3. If $\vec{t}$ satisfies (C3), it is a near false point.

Therefore, the MAX-A strategy can select $\vec{t}$. Hence, the result follows. □

Similar to the situation of double-fault expression (13), we need the following lemma in order to prove that the MAX-A strategy detects double-fault expression (19).

Lemma 5.2 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that the j_1-th and j_2-th literals, $x^{i_1}_{j_1}$ and $x^{i_1}_{j_2}$, of the i_1-th term, p_{i_1}, in S are replaced by x_{l_1} and x_{l_2}, respectively, where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 \leq k_{i_1}$, and x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}, the resulting implementation denoted by I is equivalent to that given by double-fault expression (19) in Table 1 and its detection conditions are given by (C1) to (C5) of double-fault expression (19) in Table 2. If detection conditions (C1) to (C4) cannot be satisfied, any test case satisfying detection condition (C5) is a near false point of S provided that $S \not\equiv I$.*

Proof : Since $S \not\equiv I$, there exists a test case that can satisfy any one of the following detection conditions of I as given in Table 2 for double-fault expression (19):

(C1) any point in $UTP_{i_1}(S)$ such that $x_{l_1} = 0$,

(C2) any point in $UTP_{i_1}(S)$ such that $x_{l_2} = 0$,

(C3) any point in $NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1}x_{l_2} = 1$,

(C4) any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1}x_{l_2} = 1$, or

(C5) any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_1}x_{l_2} = 1$.

Based on the assumption of the lemma, (C1) to (C4) cannot be satisfied. The test case satisfies (C5). In other words, the set of points that satisfy (C5) is non-empty.

Let $\vec{t}$ be a test case that satisfies (C5). We now prove that $\vec{t}$ is a near false point of S.

We observe that $p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_1}x_{l_2}(\vec{t}) = x^{i_1}_1 \cdots \bar{x}^{i_1}_{j_1} \cdots \bar{x}^{i_1}_{j_2} \cdots x^{i_1}_{k_{i_1}} x_{l_1}x_{l_2}(\vec{t}) = 1$. Now, let $\vec{t}_1$ be such that $\vec{t}_1$ and $\vec{t}$ only differ at the truth value of the literal $x^{i_1}_{j_2}$. Then, $p_{i_1,\bar{j}_1}x_{l_1}x_{l_2}(\vec{t}_1) = x^{i_1}_1 \cdots \bar{x}^{i_1}_{j_1} \cdots x^{i_1}_{j_2} \cdots x^{i_1}_{k_{i_1}} x_{l_1}x_{l_2}(\vec{t}_1)$ $= 1$.

Please note that $\vec{t}_1$ is a true point of S. Suppose on the contrary that $S(\vec{t}_1) = 0$. In that case, $p_{i_1,\bar{j}_1}(\vec{t}_1) = 1$ and $x_{l_1}x_{l_2}(\vec{t}_1) = 1$ because $p_{i_1,\bar{j}_1}x_{l_1}x_{l_2}(\vec{t}_1) = 1$. As a result, $\vec{t}_1 \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1}x_{l_2} = 1$. We found a test case $\vec{t}_1$ satisfying (C3). This violates the assumption that (C3) cannot be satisfied.

Since $p_{i_1,\bar{j}_1}(\vec{t}_1) = 1$, $p_{i_1}(\vec{t}_1) = 0$. In other words, $\vec{t}_1$ is a true point of S but not a true point for p_{i_1}. Hence, there exists $i_3(\neq i_1)$ such that $\vec{t}_1$ is a true point of p_{i_3} (that is, $p_{i_3}(\vec{t}_1) = 1$).

Please note that $\bar{x}_{l_1}$ does not appear in p_{i_3}. Otherwise, $p_{i_3}(\vec{t}_1) = \cdots\bar{x}_{l_1}\cdots = 0$ because $x_{l_1}x_{l_2}(\vec{t}_1) = 1$. Similarly, $\bar{x}_{l_2}$ does not appear in p_{i_3}.

We now prove that $x^{i_1}_{j_2}$ exists in p_{i_3}. First, if both $x^{i_1}_{j_2}$ and $\bar{x}^{i_1}_{j_2}$ do not appear in p_{i_3}, then $p_{i_3}(\vec{t}) = p_{i_3}(\vec{t}_1) = 1$ because $\vec{t}$ and $\vec{t}_1$ only differ at $x^{i_1}_{j_2}$. Hence, $\vec{t}$ is a true point of S. This contradicts to the fact that $\vec{t}$ is a false point of S. Second, if $\bar{x}^{i_1}_{j_2}$ appears in p_{i_3}, then $p_{i_3}(\vec{t}_1) = \cdots\bar{x}^{i_1}_{j_2}\cdots = 0$ because $p_{i_1,\bar{j}_1}(\vec{t}_1) = 1$ implies $x^{i_1}_{j_2}(\vec{t}_1) = 1$. This contradicts to the fact that $\vec{t}_1$ is a true point of p_{i_3} (that is, $p_{i_3}(\vec{t}_1) = 1$). Therefore, $x^{i_1}_{j_2}$ must appear in p_{i_3}.

Since $x^{i_1}_{j_2}$ is a literal of p_{i_3}, there is j_3 such that $x^{i_1}_{j_2}$ is the j_3-th literal of p_{i_3} (that is, $x^{i_3}_{j_3} = x^{i_1}_{j_2}$). Now, since $\vec{t}_1$ and $\vec{t}$ only differ at $x^{i_3}_{j_3}$ ($= x^{i_1}_{j_2}$) and $p_{i_3}(\vec{t}_1) = 1$, $p_{i_3,\bar{j}_3}(\vec{t}) = x^{i_3}_1\cdots\bar{x}^{i_3}_{j_3}\cdots x^{i_3}_{k_{i_3}}(\vec{t}) = 1$. In summary, $\vec{t}$ is a false point of S and $p_{i_3,\bar{j}_3}(\vec{t}) = 1$. Thus, $\vec{t} \in NFP_{i_3,\bar{j}_3}(S)$. Hence, the result follows. □

We prove Theorem 5.9 which states that the MAX-A strategy detects double-fault expression (19).

Theorem 5.9 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that the j_1-th and j_2-th literals, $x^{i_1}_{j_1}$ and $x^{i_1}_{j_2}$, of the i_1-th term, p_{i_1}, in S are replaced by x_{l_1} and x_{l_2}, respectively, where $1 \le i_1 \le m$, $1 \le j_1 < j_2 \le k_{i_1}$, and x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}, the resulting implementation denoted by I is equivalent to that given by double-fault expression (19) in Table 1 and its detection conditions are given by (C1) to (C5) of double-fault expression (19) in Table 2. The MAX-A strategy can guarantee to detect double-fault expression (19) provided that $S \not\equiv I$.*

Proof : Detection conditions (C1) to (C5) of double-fault expression (19) are listed below

(C1) any point in $UTP_{i_1}(S)$ such that $x_{l_1} = 0$,

(C2) any point in $UTP_{i_1}(S)$ such that $x_{l_2} = 0$,

(C3) any point in $NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1} = 1$,

(C4) any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or

(C5) any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} x_{l_1} x_{l_2} = 1$.

We proceed our proof using the following two cases

Case 1. Any one of the conditions (C1) to (C4) can be satisfied. Since $S \not\equiv I$, there is a test case $\vec{t}$ that satsifies (C1), (C2), (C3) or (C4). We then have the following four subcases

(a) If $\vec{t}$ satisfies (C1), $\vec{t}$ must be a unique true point.

(b) If $\vec{t}$ satisfies (C2), $\vec{t}$ must be a unique true point.

(c) If $\vec{t}$ satisfies (C3), $\vec{t}$ must be a near false point.

(d) If $\vec{t}$ satisfies (C4), $\vec{t}$ must be a near false point.

Hence, the MAX-A strategy can select $\vec{t}$.

Case 2. Conditions (C1) to (C4) cannot be satisfied. Since $S \not\equiv I$, there is a test case $\vec{t}$ that satisfies (C5). By Lemma 5.2, $\vec{t}$ is a near false point of S. Hence, the MAX-A strategy can select $\vec{t}$.

Hence, The result follows. □

Theorem 5.10 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Any test case selection strategy that subsumes the MAX-A meaningful impact strategy can detect double-fault expressions (1)–(13), (15), (17) and (19) in Table 1.*

Proof : By Theorems 5.6, 5.7, 5.8, and 5.9 and the definition of subsumption of test case selection strategy. □

Table 7: All points of S where $S = ab + ac + bd$

i	1	2	3
$UTP_i(S)$	1100	1010 , 1011	0101, 0111
$NFP_{i,\bar{1}}(S)$	0100, 0110	0010, 0011, 0110	0001, 0011, 1001
$NFP_{i,\bar{2}}(S)$	1000, 1001	1000, 1001	0100, 0110

$OTP(S)$ = {<u>1101</u>, 1110, <u>1111</u>} and $RFP(S)$ = {<u>0000</u>}

5.4 The MAX-B strategy

Since the MAX-B strategy subsumes the MAX-A strategy, it also can detect 16 double-fault expressions that MAX-A strategy detects, by Theorem 5.10. In this section, we illustrate that the MAX-B strategy, and hence all other strategies considered in this report, cannot detect double-fault expressions (14), (16) and (18) in Examples 5.5, 5.6, and 5.7, respectively, hence, completing the discussion on the relations shown in Figure 1.

Example 5.5 Let $S = ab + ac + bd$. Table 7 lists the sets $UTP_i(S)$ of all unique true points for every possible i, the sets $NFP_{i,\bar{j}}(S)$ of all near false points for every possible i and j pair, the set $OTP(S)$ of all overlapping true points and the set $RFP(S)$ of all remaining false points of S. Suppose that the literals $\bar{c}$ and $\bar{b}$ are inserted into the first and second terms of S, respectively, the resulting double-fault expression is equivalent to $I = ab\bar{c} + a\bar{b}c + bd$, which is a special instance of double-fault expression (14). Note that, S and I are not equivalent because S and I evaluate to 1 and 0 on 1110, respectively.

Now, let T be the set that includes (1) all unique true points in $UTP_i(S)$ for every i, (2) all near false points in $NFP_{i,\bar{j}}(S)$ for every i and j, (3) 2 overlapping true points, 1101 and 1111, and (4) 1 remaining false point, 0000. The test cases selected from $OTP(S)$ and $RFP(S)$ are underlined for ease of reference. We do not underline the test cases from $UTP_i(S)$ and $NFP_{i,\bar{j}}(S)$ because all of them are selected. It should be noted that T satisfies the MAX-B strategy because it contains all unique true points of S, all near false points of S, 2 out of the 3 overlapping true points of S and

Table 8: All points of S where $S = ab + ac + \bar{b}\bar{d} + \bar{a}d + b\bar{c}\bar{e}$

i	1	2	3	4	5
$UTP_i(S)$	11001, 11011	10110, 10111	00000, 00001, 00100, 00101, 10000, 10001	00010, 00011, 00110, 00111, 01011, 01110, 01111	01000
$NFP_{i,\bar{1}}(S)$	01001, 01100, 01101	01100, 01101	01001, 01100, 01101	10010, 10011	10010
$NFP_{i,\bar{2}}(S)$	10010, 10011	10010, 10011	10010, 10011	01001, 01100, 01101	01100
$NFP_{i,\bar{3}}(S)$	--	--	--	--	01001

$OTP(S) =${01010, 10100, <u>10101</u>, <u>11000</u>, 11010, 11100, 11101, <u>11110</u>, <u>11111</u>} and $RFP(S) = \emptyset$

1 remaining false point of S. Since S and I agree on all points in T, the MAX-B strategy cannot guarantee to distinguish S from I. ◊

Example 5.6 Let $S = ab + ac + \bar{b}\bar{d} + \bar{a}d + b\bar{c}\bar{e}$. Table 8 lists the sets $UTP_i(S)$ of all unique true points for every possible i, the sets $NFP_{i,\bar{j}}(S)$ of all near false points for every possible i and j pair, the set $OTP(S)$ of all overlapping true points and the set $RFP(S)$ of all remaining false points of S. Suppose that the literal e is inserted into the first term ab and the literal a in the second term ac is replaced by the literal d, the resulting double-fault expression is equivalent to $I = abe + cd + \bar{b}\bar{d} + \bar{a}d + b\bar{c}\bar{e}$. which is a special instance of double-fault expression (16). Note that, S and I are not equivalent because S and I evaluate to 1 and 0 on 11100, respectively.

Now, let T be the set that includes (1) all unique true points in $UTP_i(S)$ for every i, (2) all near false points in $NFP_{i,\bar{j}}(S)$ for every i and j, (3) 4 overlapping true points, 10101, 11000, 11110 and 11111, and (4) no remaining false points. It should be noted that T satisfies the MAX-B strategy because it contains all unique true points of S, all near false points of S, 4 out of 9 overlapping true points of S, and no remaining false points of S. Again, test cases in T from $OTP(S)$ are underlined for ease of reference. Since S and I agree on all points in T, the MAX-B strategy cannot guarantee to distinguish S from I. ◊

Table 9: All points of S where $S = abc + abd + cd\bar{f}\bar{f} + \bar{c}d\bar{e} + \bar{c}\bar{d}e + \bar{c}\bar{d}f$

i	1	2	3	4	5	6
$UTP_i(S)$	111001, 111011	110110, 110111	001000, 001010, 011000, 011010, 101000, 101010	000100, 000101, 010100, 010101, 100100, 100101	000010, 010010, 100010, 110010	000001, 010001, 100001, 110001
$NFP_{i,\bar{1}}(S)$	011001, 011011, 011100, 011101, 011110, 011111	010110, 010111, 011100, 011101, 011110, 011111	110000, 100000, 010000, 000000	001100, 001101, 011100, 011101, 101100, 101101	001011, 011011, 101011	001001, 001011, 011001, 011011, 101001, 101011
$NFP_{i,\bar{2}}(S)$	101001, 101011, 101100, 101101, 101110, 101111	100110, 100111, 101100, 101101, 101110, 101111	001100, 001110, 011100, 011110, 101100, 101110	000000, 010000, 100000, 110000	000110, 000111, 010110, 010111, 100110, 100111	000111, 010111, 100111
$NFP_{i,\bar{3}}(S)$	110000	110000	001001, 001011, 011001, 011011, 101001, 101011	000110, 000111, 010110, 010111, 100110, 100111	000000, 010000, 100000, 110000	000000, 010000, 100000, 110000

$OTP(S)$ = $\{$000011, 010011, 100011, 110011, $\underline{110100}$, $\underline{110101}$, 111000, 111010, 111100, 111101, $\underline{111110}$, $\underline{111111}\}$ and $RFP(S)$ = $\{\underline{001111}\}$.

Example 5.7 Let $S = abc + abd + cd\bar{f}\bar{f} + \bar{c}d\bar{e} + \bar{c}\bar{d}e + \bar{c}\bar{d}f$. Table 9 lists the sets $UTP_i(S)$ of all unique true points for every possible i, the sets $NFP_{i,\bar{j}}(S)$ of all near false points for every possible i and j pair, the set $OTP(S)$ of all overlapping true points and the set $RFP(S)$ of all remaining false points of S. Suppose that the literal c is the first term abc is replaced by the literal e and the literal d in the second term abd is replaced by the literal f, the resulting double-fault expression is equivalent to $I = abe + abf + cd\bar{f}\bar{f} + \bar{c}d\bar{e} + \bar{c}\bar{d}e + \bar{c}\bar{d}f$, which is a special instance of double-fault expression (18). Note that, S and I are not equivalent because S and I evaluate to 1 and 0 on 111100, respectively.

Now, let T be the set that includes (1) all unique true points in $UTP_i(S)$ for every i, (2) all near false points in $NFP_{i,\bar{j}}(S)$ for every i and j, (3) 4 overlapping true points, 110100, 110101, 111110 and 111111 and (4) 1 remaining false point, 001111. It should be noted that T satisfies the MAX-B strategy because it contains all unique true points of S, all near false points of S, 4 out of 12 overlapping true points of S, and 1 out of 1 remaining false point of S. Again, test cases in T from $OTP(S)$ are underlined for ease of reference. Since S and I agree on all points in T, the MAX-B strategy cannot guarantee to distinguish S from I. ◊

6 New Strategies

As discussed in previous section, existing test case selection strategies for single fault detection cannot guarantee to detect double literal faults. Therefore, new fault-based test case selection strategies are needed for the detection of double literal faults. In this section, we discuss how to propose new test case selection strategies to detect all double literal faults. There are two approaches in developing new test case selection strategies.

The first approach is to propose new test case selection strategies from scratch, that is without making use of any existing strategies for single fault detection. However, test case selection strategies developed via this approach may not guarantee to detect single faults. This is illustrated in Example 6.1

Example 6.1 Let $S = ab + cd + ef$. Table 10 lists the sets $UTP_i(S)$ of all unique true points, the sets $NFP_{i,\bar{j}}(S)$ of all near false points, the set $OTP(S)$ of all overlapping true points and the set $RFP(S)$ of all remaining false points of S . Let T be the set {110010, 110101, 001110, 011101, 010111, 101011, 010101, 011010, 100110, 101001, 100101, 101010, 011001, 010110}. Test cases in T have been underlined in Table 10 for ease of reference.

Table 10: All test cases of S where $S = ab + cd + ef$

i	1	2	3
$UTP_i(S)$	110000, 110001, <u>110010</u>, 110100, <u>110101</u>, 110110, 111000, 111001, 111010	001100, 001101, <u>001110</u>, 011100, <u>011101</u>, 011110, 101100, 101101, 101110	000011, 000111, 001011, 010011, <u>010111</u>, 011011, 100011, 100111, <u>101011</u>
$NFP_{i,\bar{1}}(S)$	010000 010001, 010010, 010100, <u>010101</u>, 010110, 011000, 011001, <u>011010</u>	000100, 000101, 000110, 010100, <u>010101</u>, 010110, 100100, 100101, <u>100110</u>	000001, 000101, 001001, 010001, <u>010101</u>, 011001, 100001, 100101, <u>101001</u>
$NFP_{i,\bar{2}}(S)$	100000 100001, 100010, 100100, <u>100101</u>, 100110, 101000, 101001, <u>101010</u>	001000, 001001, 001010, 011000, <u>011001</u>, 011010, 101000, 101001, <u>101010</u>	000010, 000110, 001010, 010010, <u>010110</u>, 011010, 100010, 100110, <u>101010</u>

$OTP(S)$ = {011111, 001111, 101111, 110011, 110111, 111011, <u>111100</u>, 111101, 111110, 111111} and $RFP(S)$ = {000000}

Now, as detailed below, test cases in T satisfy the detection conditions of all 19 double-fault expressions listed in Table 1, and hence T can detect all double literal faults in this report.

(A) For double-fault expressions (1) – (12): Since S is in irredundant disjunctive normal form, the sets $UTP_i(S)$ and $NFP_{i,\bar{j}}(S)$ are non-empty for every i and j. Now, the set T contains at least one element in $UTP_i(S)$ for every i and at least one element in $NFP_{i,\bar{j}}(S)$ for every i and j. Thus, T satisfies the detection conditions of these 12 double-fault expressions. As a result, T detects all these 12 double-fault expressions.

(B) For double-fault expressions (13), (16) and (18): As shown in Table 2, conditions (C3) of double-fault expression (13), (C4) of (16) and (C6) of (18) are the same: "any point in $NFP_{i_2,\bar{j}_2}(S)$ such that x_{l_2} = 1" where x_{l_2} is a missing literal of p_{i_2}. It should be noted that test cases in T are selected from $NFP_{i,\bar{j}}(S)$ for every i and j, and they collectively satisfy the following condition: "points from $NFP_{i,\bar{j}}(S)$ so that every missing literal x_l of p_i evaluates to 1 for every i and j". Hence, T satisfies the detection conditions of double-fault expressions (13), (16) and (18).

(C) For double-fault expression (14): The detection condition is "(C1) any point in $UTP_{i_1}(S)$ such that x_{l_1} = 0 or (C2) any point in $UTP_{i_2}(S)$ such that x_{l_2} = 0 or (C3) any point in $(TP_{i_1}(S) \cap$

$TP_{i_2}(S)) \setminus \left(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \right)$ such that $x_{l_1} + x_{l_2}$ = 0". Those test cases in T, selected from $UTP_i(S)$ for every i, collectively satisfy the condition: "points from $UTP_i(S)$ that make every missing literal x_l of p_i evaluate to 0 for every i", except for the missing literal $\bar{c}$ of the first term and $\bar{a}$ of the second term in $UTP_1(S)$ and $UTP_2(S)$, respectively. Therefore, all selected points from $UTP(S)$ can detect all double-fault expression (14), except $I = ab\bar{c} + \bar{a}cd + ef$. However, I can be detected by the test case 111100 in T which is selected from $OTP(S)$. Hence, T can detect double-fault expression (14).

(D) For double-fault expressions (15), (17) and (19): As shown in Table 2, the detection condition of double-fault expression (15), condition (C1) of (17), and (C1) of (19), are the same: "any point in $UTP_{i_1}(S)$ such that $x_{l_1}x_{l_2}$ = 0" where both x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}. It should be noted that test cases in T, which are selected from $UTP_i(S)$ for every i, collectively satisfy the condition: "points from $UTP_i(S)$ so that $x_{l_1}x_{l_2}$ evaluates to 0 where x_{l_1} and x_{l_2} are two different missing literals of p_i for every i". Hence, T can detect double-fault expressions (15), (17) and (19).

However, T cannot detect the literal insertion fault in the single-fault expression $I_1 = ab\bar{c} + cd + ef$ (where the literal $\bar{c}$ has been inserted into the first term ab of S) because S and I_1 agree on all points in T. Since S and I_1 evaluate to 1 and 0 on 111010, they are not equivalent. $\Diamond$

The second approach is to propose new test case selection strategies that can supplement existing test case selection strategies for single faults to detect double faults. Hence, when all these strategies are applied, both single and double faults can be detected. Moreover, this approach is more practical in testing than the previous approach because of the following two reasons:

1. Due to time and resource limitation, software testing practitioners may opt to select test cases using existing test case selection strategies for single faults to first ensure that the program

is free from single faults. When there is still room for further testing, they may then opt to use supplementary strategies that can help increase the chance of detecting other fault classes, such as double faults.

2. As shown in [7], some existing test case selection strategies for single fault detection can guarantee to detect certain double faults. It would be simpler to develop new strategies that focus only on the undetected double faults.

In this report, we adopt this second approach.

Before proposing new strategies, we need to identify which existing test case selection strategies can be used as the basis for detecting all single fault classes. Among the existing strategies discussed earlier, it has been shown that the BASIC strategy cannot guarantee to detect LIF and LRF [1], while the MUMCUT strategy (and hence also the MAX-A and MAX-B strategies which subsume the MUMCUT strategy) guarantees to detect all single fault classes in [1, 9]. Moreover, empirical studies [17, 18] have shown that the MUMCUT, MAX-A and MAX-B strategies require, on average, 12.0%, 40.6% and 48.2% of the entire input domain, respectively. Thus, the MUMCUT strategy is more cost-effective than the MAX-A and MAX-B strategies in detecting single faults. Furthermore, as discussed previously, the MUMCUT, the MAX-A and the MAX-B strategies cannot detect 6, 3, and 3 out of the 19 double-fault expressions, respectively. Thus, the MUMCUT strategy is almost as effective as the MAX-A and MAX-B strategies in detecting the double fault expressions considered in this report, but at a much smaller sized test sets. Hence, we choose to supplement the MUMCUT strategy for double literal fault detection. Note that although we only discuss the case for the MUMCUT strategy, the idea of supplementing other strategies is similar.

Next, as discussed previously, the MUMCUT strategy cannot guarantee to detect six out of 19 double-fault expressions, namely, expressions (13), (14), (16)–(19) as shown in Table 1. Hence, we need to propose new test case selection strategies that supplement the MUMCUT strategy to detect these 6 double-fault expressions. Table 11 indicates these 6 newly proposed strategies and their

Table 11: Supplementary strategies for double literal faults

New Strategy	Section
Supplementary Multiple False Point (SMFP) strategy	6.1
Supplementary Multiple Overlapping True Point (SMOTP) strategy	6.2
Supplementary Multiple Unique True Point (SMUTP) strategy	6.3
Pairwise Multiple Unique True Point (PMUTP) strategy	6.4
Pairwise Multiple Near False Point (PMNFP) strategy	6.5
Supplementary Pairwise Multiple False Point (SPMFP) strategy	6.6

Table 12: Test case selection strategies supplementary to the MUMCUT strategy

Expr No.	Test Case Selection Strategies
(13)	MUMCUT and SMFP strategies
(14)	MUMCUT and SMOTP strategies
(16)	MUMCUT, PMUTP, SMUTP and SMOTP strategies
(17)	MUMCUT and PMNFP strategies
(18)	MUMCUT, PMUTP, SMUTP, and SMOTP strategies
(19)	MUMCUT, PMNFP and SPMFP strategies

corresponding discussions in the rest of this section. Moreover, Table 12 indicates which strategies supplement the MUMCUT strategy for the detection of a particular double-fault expression. For example, the MUMCUT strategy together with the *Supplementary Multiple Overlapping True Point (SMOTP)* strategy can guarantee to detect double-fault expression (14). In the rest of this section, we discuss these 6 strategies and explain how they supplement the MUMCUT strategy in detecting double faults.

6.1 The SMFP Strategy

We propose the *Supplementary Multiple False Point (SMFP)* strategy to detect double-fault expression (13) in Table 1. It aims at selecting test cases satisfying the detection condition (C4) of double-fault expression (13), that is "any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2}x_{l_2} = 1$ where x_{l_2} is a missing literal of p_i", as listed in Table 2.

The SMFP strategy requires to select test cases from $FP(S)$ to form a set T such that, for every term p_i of S and for every pair of literals $x^i_{j_1}$ and $x^i_{j_2}$ in p_i and for every missing literal x_l of p_i,

1. there is a test case $\vec{t}_1 \in T$ such that $p_{i,\bar{j}_1,\bar{j}_2} = 1$ and $x_l = 1$ if possible; and
2. there is a test case $\vec{t}_2 \in T$ such that $p_{i,\bar{j}_1,\bar{j}_2} = 1$ and $x_l = 0$ if possible.

In other words, points in T are from $FP(S)$ such that (1) $p_{i,\bar{j}_1,\bar{j}_2} = 1$ for every pair of literals $x^i_{j_1}$ and $x^i_{j_2}$ in every term p_i, and (2) they cover all possible truth values of every missing literal x_l of every term p_i. Example 6.2 illustrates how to select test cases satisfying the SMFP strategy.

Example 6.2 Let $S = ab + cd + ef$. Table 13 lists the set of all false points of S. Now, let T = $\{$000101, 001010, 010001, 010100, 100010, 101000$\}$. Test cases in T are underlined in Table 13 for ease of reference. The set T satisfies the SMFP strategy because of the following reasons

(1) For the first term ab of S, the test cases 000101 and 001010 can make $\bar{a}\bar{b} = 1$ and they cover all possible truth values of every missing literal (c, d, e and f) of ab. Hence, they satisfy the requirements of the SMFP strategy for the first term of S.

(2) Similarly, test cases 010001 and 100010 satisfy the requirements of the SMFP strategy for the second term of S, test cases 010100 and 101000 for the third term of S.

◊

Table 13: All false points of S where $S = ab + cd + ef$

000000, 000001, 000010, 000100, 000101, 000110, 001000, 001001, 001010, 010000, 010001, 010010, 010100, 010101, 010110, 011000, 011001, 011010, 100000, 100001, 100010, 100100, 100101, 100110, 101000, 101001, 101010

We prove in Theorem 6.1 that the SMFP and MUMCUT strategies can select test cases to detect double-fault expression (13).

Theorem 6.1 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that the j_1-th literal, $x^{i_1}_{j_1}$, of the i_1-th term, p_{i_1}, in S is omitted and the j_2-th literal, $x^{i_1}_{j_2}$, of p_{i_1} is replaced by x_{l_2}, where $1 \le i_1 \le m$, $1 \le j_1 < j_2 \le k_{i_1}$, k_{i_1} is the number of literals in p_{i_1} and x_{l_2} is a missing literal of p_{i_1}, the resulting implementation denoted as I will be equivalent to that given by double-fault expression (13) in Table 1 and its detection conditions are given by the corresponding conditions (C1) to (C4) of double-fault expression (13) in Table 2. The SMFP strategy can supplement the MUMCUT strategy to guarantee the detection of double-fault expression (13) provided that $S \not\equiv I$.*

Proof : Since S and I are not equivalent, by Theorem 4.13, there is a point $\vec{t}_1$ that satisfies any of the following conditions

(C1) $\vec{t}_1 \in UTP_{i_1}(S)$ such that $x_{l_2} = 0$

(C2) $\vec{t}_1 \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_2} = 1$

(C3) $\vec{t}_1 \in NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or

(C4) $\vec{t}_1 \in FP(S)$ such that $p_{i,\bar{j}_1,\bar{j}_2} x_{l_2} = 1$.

We then have the following cases:

Case 1 $\vec{t}_1$ satisfies (C1). Then, $\{\vec{t} \in UTP_{i_1}(S) : x_{l_2}(\vec{t}) = 0\} \ne \emptyset$. Since x_{l_2} is a missing literal of p_{i_1}, the MUTP strategy can select a point that satisfies (C1).

Case 2 $\vec{t}_1$ satisfies (C2). Then, $\{\vec{t} \in NFP_{i_1,\bar{j}_1}(S) : x_{l_2}(\vec{t}) = 1\} \neq \emptyset$. Since x_{l_2} is a missing literal of p_{i_1}, the MNFP strategy can select a point that satisfies (C2).

Case 3 $\vec{t}_1$ satisfies (C3). Then, $\{\vec{t} \in NFP_{i_1,\bar{j}_2}(S) : x_{l_2}(\vec{t}) = 1\} \neq \emptyset$. Since x_{l_2} is a missing literal of p_{i_1}, the MNFP strategy can select a point that satisfies (C3).

Case 4 $\vec{t}_1$ satisfies (C4). Then, $\{\vec{t} \in FP(S) : p_{i,\bar{j}_1,\bar{j}_2} x_{l_2}(\vec{t}) = 1\} = \{\vec{t} \in FP(S) : p_{i,\bar{j}_1,\bar{j}_2}(\vec{t}) = 1$ *and* $x_{l_2}(\vec{t}) = 1\} \neq \emptyset$. The SMFP strategy can select a point that satisfies (C4).

Hence, the result follows. □

6.2 The Supplementary Multiple Overlapping True Point (SMOTP) Strategy

We propose the *Supplementary Multiple Overlapping True Point (SMOTP)* strategy to detect double-fault expression (14) in Table 1. In fact, it aims at selecting test cases that satisfy the following detection conditions

1. (C3) of double-fault expression (14) (that is, "any point in $\left(\left(TP_{i_1}(S) \cap TP_{i_2}(S)\right) \setminus \left(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S)\right)\right)$ such that $x_{l_1} + x_{l_2} = 0$"),
2. (C5) of double-fault expression (16) (that is, "any point in $\left(\left(TP_{i_1}(S) \cap TP_{i_2}(S)\right) \setminus \left(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S)\right)\right)$ such that $x_{l_1} + x_{l_2} = 0$"), and
3. (C7) of double-fault expression (18) (that is, "any point in $\left(\left(TP_{i_1}(S) \cap TP_{i_2}(S)\right) \setminus \left(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S)\right)\right)$ such that $x_{l_1} + x_{l_2} = 0$").

The SMOTP strategy requires to select test cases from $\left(TP_{i_1}(S) \cap TP_{i_2}(S)\right) \setminus \left(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S)\right)$, for every two different terms p_{i_1} and p_{i_2} of S, to form a set T such that, for every missing literal x_{l_1} of p_{i_1} and every missing literal x_{l_2} of p_{i_2},

1. there is a test case $\vec{t}_1 \in T$ such that $x_{l_1} = 0$ and $x_{l_2} = 0$, if possible;
2. there is a test case $\vec{t}_2 \in T$ such that $x_{l_1} = 0$ and $x_{l_2} = 1$, if possible;
3. there is a test case $\vec{t}_3 \in T$ such that $x_{l_1} = 1$ and $x_{l_2} = 0$, if possible; and
4. there is a test case $\vec{t}_4 \in T$ such that $x_{l_1} = 1$ and $x_{l_2} = 1$, if possible.

In other words, points in T are from $\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$ for every pair of different terms p_{i_1} and p_{i_2} of S such that they can cover all possible truth value combinations of x_{l_1} and x_{l_2} (that is 00, 01, 10 and 11) for every missing literal x_{l_1} of p_{i_1} and every missing literal x_{l_2} of p_{i_2}. Example 6.3 illustrates how to select test cases satisfying the SMOTP strategy.

Example 6.3 Let $S = ab + cd + e$. The set of overlapping true points of S is $\{00111, 01111, 10111, 11001, 11011, 11101, 11110, 11111\}$. Now, let $T = \{00111, 01111, 10111, 11001, 11011, 11101, 11110\}$. The set T satisfies the SMOTP strategy because of the following reasons

(1) For the first two terms ab and cd of S, $\Big(\big(TP_1(S) \cap TP_2(S)\big) \setminus TP_3(S)\Big) = \{11110\}$. This test case, 11110, covers all possible truth value combinations of every possible pair of missing literals x_{l_1} of ab and x_{l_2} of cd. Hence, 11110 satisfies the requirements of the SMOTP strategy on $\Big(\big(TP_1(S) \cap TP_2(S)\big) \setminus TP_3(S)\Big)$.

(2) For the first term ab and third term e of S, $\Big(\big(TP_1(S) \cap TP_3(S)\big) \setminus TP_2(S)\Big) = \{11001, 11011, 11101\}$. The test cases 11001, 11011, and 11101 can cover all possible truth value combinations of every possible pair of missing literals x_{l_1} of ab and x_{l_2} of e. Thus, they satisfy the requirements of the SMOTP strategy on $\Big(\big(TP_1(S) \cap TP_3(S)\big) \setminus TP_2(S)\Big)$.

(3) Similarly, test cases 00111, 01111 and 10111 cover all possible truth value combinations of every possible pair of missing literals x_{l_1} of cd and x_{l_2} of e. Thus, they satisfy the requirements of the SMOTP strategy on $\Big(\big(TP_2(S) \cap TP_3(S)\big) \setminus TP_1(S)\Big)$. ◊

We now prove that the SMOTP strategy supplements the MUMCUT strategy to guarantee the detection of double-fault expression (14). Since extra strategies are required to detect double-fault expressions (16) and (18), we will postpone the discussion to later sections.

Theorem 6.2 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that two literals x_{l_1} and x_{l_2} are inserted into the i_1-th term, p_{i_1}, and the i_2-th term, p_{i_2}, in S, respectively, where $1 \leq i_1 < i_2 \leq m$, x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively, the resulting implementation denoted as I will be equivalent to that given by double-fault expression (14) in Table 1 and its detection conditions are given by the corresponding conditions (C1) to (C3) of double-fault expression (14) in Table 2. The SMOTP strategy can supplement the MUMCUT strategy to guarantee the detection of double-fault expression (14) provided that $S \not\equiv I$.*

Proof : Since S and I are not equivalent, by Theorem 4.14, there is a point $\vec{t}_1$ that satisfies any of the following conditions

(C1) $\vec{t}_1 \in UTP_{i_1}(S)$ such that $x_{l_1} = 0$

(C2) $\vec{t}_1 \in UTP_{i_2}(S)$ such that $x_{l_2} = 0$, or

(C3) $\vec{t}_1 \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$ such that $x_{l_1} + x_{l_2} = 0$.

We then have the following cases:

Case 1 $\vec{t}_1$ satisfies (C1). Then, $\{\vec{t} \in UTP_{i_1}(S) : x_{l_1}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_1} is a missing literal of p_{i_1}, the MUTP strategy can select a point that satisfies (C1).

Case 2 $\vec{t}_1$ satisfies (C2). Then, $\{\vec{t} \in UTP_{i_2}(S) : x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_2} is a missing literal of p_{i_2}, the MUTP strategy can select a point that satisfies (C2).

Case 3 $\vec{t}_1$ satisfies (C3). Then, $\{\vec{t} \in \left(TP_{i_1}(S) \cap TP_{i_2}(S)\right) \setminus \left(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\right) : x_{l_1}(\vec{t}) + x_{l_2}(\vec{t}) = 0\} =$

$\{\vec{t} \in \left(TP_{i_1}(S) \cap TP_{i_2}(S)\right) \setminus \left(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\right) : x_{l_1}(\vec{t}) = 0 \text{ and } x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. The SMOTP strategy can select a point that satisfies (C3).

Hence, the result follows. □

6.3 The Supplementary Multiple Unique True Point (SMUTP) Strategy

We propose the *Supplementary Multiple Unique True Point (SMUTP)* strategy to select test cases that satisfy the following detection conditions

1. (C1) of double-fault expression (16) (that is, "any point in $UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$" where x_{l_1} is a missing literal of p_{i_1} and $i_1 \neq i_2$),
2. (C1) of double-fault expression (18) (that is, "any point in $UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$" where x_{l_1} is a missing literal of p_{i_1} and $i_1 \neq i_2$), and
3. (C3) of double-fault expression (18) (that is, "any point in $UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$" where x_{l_2} is a missing literal of p_{i_2} and $i_1 \neq i_2$).

The SMUTP strategy requires to select test cases from $UTP_{i_1}(S)$, for every term p_{i_1} of S, to form a set T such that, for every term p_{i_2} ($i_1 \neq i_2$), every literal $x^{i_2}_{j_2}$ in p_{i_2} and every missing literal x_{l_1} of p_{i_1},

1. there is a test case $\vec{t}_1 \in T$ such that $p_{i_2,\hat{j}_2} = 0$ and $x_{l_1} = 0$, if possible; and
2. there is a test case $\vec{t}_2 \in T$ such that $p_{i_2,\hat{j}_2} = 0$ and $x_{l_1} = 1$, if possible.

Table 14: All unique true points of S where $S = abc + def$

$UTP_1(S)$	111000, 111001, 111010, 111011, 111100, 111101, 111110
$UTP_2(S)$	000111, 001111, 010111, 011111, 100111, 101111, 110111

In other words, points in T are from $UTP_{i_1}(S)$ for every p_{i_1} such that (1) $p_{i_2,\hat{j}_2}$ = 0 for every term p_{i_2} different from p_{i_1} and every literal $x^{i_2}_{j_2}$ in p_{i_2}, and (2) they cover all possible truth values of x_{l_1} for every missing literal x_{l_1} of p_{i_1}. Example 6.4 illustrates how to select test cases satisfying the SMUTP strategy.

Example 6.4 Let $S = abc + def$. Table 14 lists the sets $UTP_i(S)$ of unique true points of S. Now, let T = {111010, 111011, 111100, 111101, 010111, 011111, 100111, 101111}. Test cases in T are underlined in Table 14 for ease of reference. Test cases in T satisfy the SMUTP strategy because of the following reasons

(1) For the first term abc of S, we need to select test cases from $UTP_1(S)$. In $UTP_1(S)$, the test cases 111010 and 111101 are such that $p_{2,\hat{1}} = ef$ = 0 and they cover all possible truth values of every missing literal (d, e and f) of $p_1 = abc$.

    Similarly, the test cases 111011 and 111100 in $UTP_1(S)$ are such that $p_{2,\hat{2}} = df$ = 0 and they cover all possible truth values of every missing literal (d, e and f) of abc.

    Finally, the test cases 111010 and 111101, previously selected for $p_{2,\hat{1}} = ef = 0$, are such that $p_{2,\hat{3}} = de$ = 0 and they cover all possible truth values of every missing literal (d, e and f) of abc. Incidentally, the test cases 111011 and 111100, previously selected for $p_{2,\hat{2}} = df$ = 0, also cause $p_{2,\hat{3}} = de$ = 0 and cover all possible truth values of every missing literal (d, e and f) of abc. Hence, these four test cases satisfy the requirements of the SMUTP strategy on $UTP_1(S)$.

(2) Similarly, test cases 010111, 011111, 100111 and 101111 satisfy the requirements of the SMUTP strategy on $UTP_2(S)$. ◊

To guarantee the detection of double-fault expressions (16) and (18), we need an extra strategy which is discussed in the next section.

6.4 The Pairwise Multiple Unique True Point (PMUTP) Strategy

We propose the *Pairwise Multiple Unique True Point (PMUTP)* strategy to select test cases that satisfy the following detection conditions

1. (C2) of double-fault expression (16) (that is, "any point in $UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2}$ = 0" where x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively),
2. (C2) of double-fault expression (18) (that is "any point in $UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2}$ = 0" where x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively), and
3. (C4) of double-fault expression (18) (that is, "any point in $UTP_{i_2}(S)$ such that $x_{l_1} + x_{l_2}$ = 0" where x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively).

The PMUTP strategy requires to select test cases from every $UTP_{i_1}(S)$ for every p_{i_1} of S to form a set T such that, for every term p_{i_2} different from p_{i_1} (that is, $i_2 \neq i_1$), for every missing literal x_{l_1} of p_{i_1}, and every missing literal x_{l_2} of p_{i_2},

1. there is a test case $\vec{t}_1 \in T$ such that x_{l_1} = 0 and x_{l_2} = 0, if possible;
2. there is a test case $\vec{t}_2 \in T$ such that x_{l_1} = 0 and x_{l_2} = 1, if possible;
3. there is a test case $\vec{t}_3 \in T$ such that x_{l_1} = 1 and x_{l_2} = 0, if possible; and
4. there is a test case $\vec{t}_4 \in T$ such that x_{l_1} = 1 and x_{l_2} = 1, if possible.

In other words, points in T are from $UTP_{i_1}(S)$ for every p_{i_1} such that, for every term p_{i_2} different from p_{i_1}, they can cover all possible truth value combinations of x_{l_1} and x_{l_2} (that is 00, 01, 10 and

Table 15: All unique true points of S where $S = ab + cd + ef$

$UTP_1(S)$	110000, 110001, 110010, 110100, 110101, 110110, 111000, 111001, 111010
$UTP_2(S)$	001100, 001101, 001110, 011100, 011101, 011110, 101100, 101101, 101110
$UTP_3(S)$	000011, 000111, 001011, 010011, 010111, 011011, 100011, 100111, 101011

11) for every missing literal x_{l_1} of p_{i_1} and every missing literal x_{l_2} of p_{i_2}. Example 6.5 illustrates how to select test cases satisfying the PMUTP strategy.

Example 6.5 Let $S = ab + cd + ef$. Table 15 lists the sets $UTP_i(S)$ of all unique true points of S. Now, let T = {110000, 110101, 110110, 111001, 111010, 001100, 011101, 011110, 101101, 101110, 000011, 010111, 011011, 100111, 101011}. Test cases in T are underlined in Table 15 for ease of reference. Test cases in T satisfy the PMUTP strategy because of the following reasons

(1) For the first term ab, the test cases 110000, 110101, 110110, 111001 and 111010 cover all possible truth value combinations of every possible pair of missing literals x_{l_1} of ab (c, d, e and f) and x_{l_2} of cd (a, b, e and f). It should also be noted that it is impossible to select points from $UTP_1(S)$ such that both e and f evaluate to 1 because such points are true points of $p_3 = ef$ and, hence, are not unique true points of p_1.

Furthermore, the previously selected 5 test cases also cover all possible truth value combinations of every possible pair of missing literals x_{l_1} of ab and x_{l_2} of ef.

As a result, these 5 test cases satisfy the requirements of PMUTP strategy on $UTP_1(S)$.

(2) Similarly, the test cases 001100, 011101, 011110, 101101 and 101110 from $UTP_2(S)$ cover all possible truth value combinations of every possible pair of missing literals x_{l_1} of cd and x_{l_2} of ab.

Furthermore, they cover all possible truth value combinations of every possible pair of missing literals x_{l_1} of cd and x_{l_2} of ef.

Hence, these 5 test cases satisfy the requirements of PMUTP strategy on $UTP_2(S)$.

(3) Finally, the test cases 000011, 010111, 011011, 100111 and 101011 satisfy the requirements of PMUTP strategy on $UTP_3(S)$. $\Diamond$

We now prove that the SMUTP, PMUTP, SMOTP and MUMCUT strategies together can select test cases to detect double-fault expressions (16) and (18) in Theorems 6.3 and 6.4, respectively.

Theorem 6.3 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that the literal x_{l_1} is inserted into the i_1-th term, p_{i_1}, in S and the j_2-th literal, $x^{i_2}_{j_2}$, of the i_2-th term, p_{i_2}, in S is replaced by x_{l_2}, where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_2 \leq k_{i_2}$, k_{i_2} is the number of literals in p_{i_2}, and x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively, the resulting implementation denoted by I will be equivalent to that given by double-fault expression (16) in Table 1 and its detection conditions are given by the corresponding conditions (C1) to (C5) of double-fault expression (16) in Table 2. The SMUTP, PMUTP and SMOTP strategies can supplement the MUMCUT strategy to guarantee the detection of double-fault expression (16) provided that $S \not\equiv I$.*

Proof : Since S and I are not equivalent, by Theorem 4.16, there is a point $\vec{t_1}$ that satisfies any of the following conditions

(C1) $\vec{t_1} \in UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$,

(C2) $\vec{t_1} \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

(C3) $\vec{t_1} \in UTP_{i_2}(S)$ such that $x_{l_2} = 0$,

(C4) $\vec{t_1} \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or

(C5) $\vec{t_1} \in \Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)\Big)$ such that $x_{l_1} + x_{l_2} = 0$.

We then have the following cases:

Case 1 $\vec{t}_1$ satisfies (C1). Then, $\{\vec{t} \in UTP_{i_1}(S) : p_{i_2,\hat{j}_2}(\vec{t}) + x_{l_1}(\vec{t}) = 0\} = \{\vec{t} \in UTP_{i_1}(S) : p_{i_2,\hat{j}_2}(\vec{t}) = 0 \text{ and } x_{l_1}(\vec{t}) = 0\} \neq \emptyset$. The SMUTP strategy can select a point that satisfies (C1).

Case 2 $\vec{t}_1$ satisfies (C2). Then, $\{\vec{t} \in UTP_{i_1}(S) : x_{l_1}(\vec{t}) + x_{l_2}(\vec{t}) = 0\} = \{\vec{t} \in UTP_{i_1}(S) : x_{l_1}(\vec{t}) = 0 \text{ and } x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively, the PMUTP strategy can select a point that satisfies (C2).

Case 3 $\vec{t}_1$ satisfies (C3). Then, $\{\vec{t} \in UTP_{i_2}(S) : x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_2} is a missing literal of p_{i_2}, the MUTP strategy can select a point that satisfies (C3).

Case 4 $\vec{t}_1$ satisfies (C4). Then, $\{\vec{t} \in NFP_{i_2,\bar{j}_2}(S) : x_{l_2}(\vec{t}) = 1\} \neq \emptyset$. Since x_{l_2} is a missing literal of p_{i_2}, the MNFP strategy can select a point that satisfies (C4).

Case 5 $\vec{t}_1$ satisfies (C5). Then, $\{\vec{t} \in \Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,i_2}}^{m} TP_i(S)\big)\Big) : x_{l_1}(\vec{t}) + x_{l_2}(\vec{t}) = 0\} = \{\vec{t} \in \Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,i_2}}^{m} TP_i(S)\big)\Big) : x_{l_1}(\vec{t}) = 0 \text{ and } x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively, the SMOTP strategy can select a point that satisfies (C5).

Hence, the result follows. □

Theorem 6.4 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that the j_1-th literal, $x^{i_1}_{j_1}$, of the i_1-th term, p_{i_1}, in S is replaced by x_{l_1} and the j_2-th literal, $x^{i_2}_{j_2}$, of the i_2-th term, p_{i_2}, in S is replaced by x_{l_2} where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_1 \leq k_{i_1}$, $1 \leq j_2 \leq k_{i_2}$, k_{i_1} and k_{i_2} are the numbers of literals in p_{i_1} and p_{i_2}, respectively, and x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively, the resulting implementation denoted by I will be equivalent to that given by double-fault expression (18) in Table 1 and its detection conditions are given by the corresponding conditions (C1) to (C7) of double-fault expression (18) in Table 2. The SMUTP, PMUTP and SMOTP strategies can supplement the MUMCUT strategy to guarantee the detection of double-fault expression (18) provided that $S \not\equiv I$.*

Proof : Since S and I are not equivalent, by Theorem 4.18, there is a point $\vec{t}_1$ that satisfies any of the following conditions

(C1) $\vec{t}_1 \in UTP_{i_1}(S)$ such that $p_{i_2,\hat{j}_2} + x_{l_1} = 0$,

(C2) $\vec{t}_1 \in UTP_{i_1}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

(C3) $\vec{t}_1 \in UTP_{i_2}(S)$ such that $p_{i_1,\hat{j}_1} + x_{l_2} = 0$,

(C4) $\vec{t}_1 \in UTP_{i_2}(S)$ such that $x_{l_1} + x_{l_2} = 0$,

(C5) $\vec{t}_1 \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1} = 1$,

(C6) $\vec{t}_1 \in NFP_{i_2,\bar{j}_2}(S)$ such that $x_{l_2} = 1$, or

(C7) $\vec{t}_1 \in \Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S)\big)\Big)$ such that $x_{l_1} + x_{l_2} = 0$.

We then have the following cases:

Case 1 $\vec{t}_1$ satisfies (C1). Then, $\{\vec{t} \in UTP_{i_1}(S) : p_{i_2,\hat{j}_2}(\vec{t}) + x_{l_1}(\vec{t}) = 0\} = \{\vec{t} \in UTP_{i_1}(S) : p_{i_2,\hat{j}_2}(\vec{t}) = 0 \textit{ and } x_{l_1}(\vec{t}) = 0\} \neq \emptyset$. The SMUTP strategy can select a point that satisfies (C1).

Case 2 $\vec{t}_1$ satisfies (C2). Then, $\{\vec{t} \in UTP_{i_1}(S) : x_{l_1}(\vec{t}) + x_{l_2}(\vec{t}) = 0\} = \{\vec{t} \in UTP_{i_1}(S) : x_{l_1}(\vec{t}) = 0 \textit{ and } x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively, the PMUTP strategy can select a point that satisfies (C2).

Case 3 $\vec{t}_1$ satisfies (C3). Then, $\{\vec{t} \in UTP_{i_2}(S) : p_{i_1,\hat{j}_1}(\vec{t}) + x_{l_2}(\vec{t}) = 0\} = \{\vec{t} \in UTP_{i_2}(S) : p_{i_1,\hat{j}_1}(\vec{t}) = 0 \textit{ and } x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. The SMUTP strategy can select a point that satisfies (C3).

Case 4 $\vec{t}_1$ satisfies (C4). Then, $\{\vec{t} \in UTP_{i_2}(S) : x_{l_1}(\vec{t}) + x_{l_2}(\vec{t}) = 0\} = \{\vec{t} \in UTP_{i_2}(S) : x_{l_1}(\vec{t}) = 0 \textit{ and } x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively, the PMUTP strategy can select a point that satisfies (C4).

Case 5 $\vec{t}_1$ satisfies (C5). Then, $\{\vec{t} \in NFP_{i_1,\bar{j}_1}(S) : x_{l_1}(\vec{t}) = 1\} \neq \emptyset$. Since x_{l_1} is a missing literal of p_{i_1}, the MNFP strategy can select a point that satisfies (C5).

Case 6 $\vec{t}_1$ satisfies (C6). Then, $\{\vec{t} \in NFP_{i_2,\bar{j}_2}(S) : x_{l_2}(\vec{t}) = 1\} \neq \emptyset$. Since x_{l_2} is a missing literal of p_{i_2}, the MNFP strategy can select a point that satisfies (C6).

Case 7 $\vec{t}_1$ satisfies (C7). Then, $\{\vec{t} \in \Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1,i_2}}^{m} TP_i(S)\big)\Big) : x_{l_1}(\vec{t}) + x_{l_2}(\vec{t}) = 0\} = \{\vec{t} \in \Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1,i_2}}^{m} TP_i(S)\big)\Big) : x_{l_1}(\vec{t}) = 0 \textit{ and } x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_1} and x_{l_2} are missing literals of p_{i_1} and p_{i_2}, respectively, the SMOTP strategy can select a point that satisfies (C7).

Hence, the result follows. □

6.5 The Pairwise Multiple Near False Point (PMNFP) Strategy

We propose the *Pairwise Multiple Near False Point (PMNFP)* strategy to detect double-fault expression (17) in Table 1. It aims at selecting test cases that satisfy the following detection conditions

1. (C2) of double-fault expression (17) (that is, "any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1}x_{l_2}$=1" where x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}),
2. (C2) of double-fault expression (19) (that is, "any point in $NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1}x_{l_2}$=1" where x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}), and
3. (C3) of double-fault expression (19) (that is, "any point in $NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1}x_{l_2}$=1" where x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}).

The PMNFP strategy requires to select test cases from every possible $NFP_{i,\bar{j}}(S)$ of S to form a set T such that, for every pair of missing literals x_{l_1} and x_{l_2} of p_i where $x_{l_1} \neq x_{l_2}$ and $x_{l_1} \neq \bar{x}_{l_2}$,

Table 16: All near false points of S where $S = ab + cd + ef$

i	1	2	3
$NFP_{i,\bar{1}}(S)$	010000, 010001, 010010, 010100, 010101, 010110, 011000, 011001, 011010	000100, 000101, 000110, 010100, 010101, 010110, 100100, 100101, 100110	000001, 000101, 001001, 010001, 010101, 011001, 100001, 100101, 101001
$NFP_{i,\bar{2}}(S)$	100000, 100001, 100010, 100100, 100101, 100110, 101000, 101001, 101010,	001000, 001001, 001010, 011000, 011001, 011010, 101000, 101001, 101010	000010, 000110, 001010, 010010, 010110, 011010, 100010, 100110, 101010

1. there is a test case $\vec{t}_1 \in T$ such that x_{l_1} = 0 and x_{l_2} = 0, if possible;
2. there is a test case $\vec{t}_2 \in T$ such that x_{l_1} = 0 and x_{l_2} = 1, if possible;
3. there is a test case $\vec{t}_3 \in T$ such that x_{l_1} = 1 and x_{l_2} = 0, if possible; and
4. there is a test case $\vec{t}_4 \in T$ such that x_{l_1} = 1 and x_{l_2} = 1, if possible.

In other words, points in T are from $NFP_{i,\bar{j}}(S)$ for every p_i of S such that, they can cover all possible truth value combinations of x_{l_1} and x_{l_2} (that is 00, 01, 10 and 11) for every pair of two different missing literals x_{l_1} and x_{l_2} of p_i. Example 6.6 illustrates how to select test cases satisfying the PMNFP strategy.

Example 6.6 Let $S = ab + cd + ef$. Table 16 lists the sets $NFP_{i,\bar{j}}(S)$ of all near false points of S. Now, let T = {010000, 010101, 010110, 011001, 011010, 000100, 010101, 010110, 100101, 100110, 000001, 010101, 011001, 100101, 101001, 100000, 100101, 100110, 101001, 101010, 001000, 011001, 011010, 101001, 101010, 000010, 010110, 011010, 100110, 101010}. Test cases in T are underlined in Table 16 for ease of references. Test cases in T satisfy the PMNFP strategy because of the following reasons

(1) For $NFP_{1,\bar{1}}(S)$, the test cases 010000, 010101, 010110, 011001 and 011010 cover all possible truth value combinations of every possible pair of missing literals (c, d, e and f) of $p_1 = ab$. It

should be noted that it is impossible to select points from $NFP_{1,\bar{1}}(S)$ such that both c and d evaluate to 1 because such points are true points of $p_2 = cd$ and, hence, are true points of S. Hence, these five test cases satisfy the requirements of the PMNFP strategy on $NFP_{1,\bar{1}}(S)$.

(2) Similarly, test cases underlined in each $NFP_{i,\bar{j}}(S)$ in Table 16 cover all possible truth value combinations of every possible pair of missing literals of every term p_i. $\Diamond$

We now prove that the PMNFP and MUMCUT strategies can select test cases to detect double-fault expression (17) in Theorem 6.5.

Theorem 6.5 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that the literal x_{l_1} is inserted into the i_1-th term, p_{i_1}, in S and the j_2-th literal, $x^{i_1}_{j_2}$, of p_{i_1}, in S is replaced by x_{l_2} where $1 \leq i_1 \leq m$, $1 \leq j_2 \leq k_{i_1}$, k_{i_1} is the number of literals in p_{i_1}, and x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}, the resulting implementation denoted by I will be equivalent to that given by double-fault expression (17) in Table 1 and its detection conditions are given by the corresponding conditions (C1) to (C3) of double-fault expression (17) in Table 2. The PMNFP strategy can supplement the MUMCUT strategy to guarantee the detection of double-fault expression (17) provided that $S \not\equiv I$.*

Proof : Since S and I are not equivalent, by Theorem 4.17, there is a point $\vec{t}_1$ that satisfies any of the following conditions

(C1) $\vec{t}_1 \in UTP_{i_1}(S)$ such that x_{l_1} = 0,

(C2) $\vec{t}_1 \in UTP_{i_1}(S)$ such that x_{l_2} = 0, or

(C3) $\vec{t}_1 \in NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1}x_{l_2}$ = 1.

We then have the following cases:

Case 1 $\vec{t}_1$ satisfies (C1). Then, $\{\vec{t} \in UTP_{i_1}(S) : x_{l_1}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_1} is a missing literal of p_{i_1}, the MUTP strategy can select a point that satisfies (C1).

Case 2 $\vec{t}_1$ satisfies (C2). Then, $\{\vec{t} \in UTP_{i_1}(S) : x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_2} is a missing literal of p_{i_1}, the MUTP strategy can select a point that satisfies (C2).

Case 3 $\vec{t}_1$ satisfies (C3). Then, $\{\vec{t} \in NFP_{i_1,\bar{j}_2}(S) : x_{l_1}x_{l_2}(\vec{t}) = 1\} = \{\vec{t} \in NFP_{i_1,\bar{j}_2}(S) : x_{l_1}(\vec{t}) = 1$ *and* $x_{l_2}(\vec{t}) = 1\} \neq \emptyset$. Since x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}, the PMNFP strategy can select a point that satisfies (C3).

Hence, the result follows. □

6.6 The Supplementary Pairwise Multiple False Point (SPMFP) Strategy

We propose the *Supplementary Pairwise Multiple False Point (SPMFP)* strategy to select test cases satisfying the detection condition (C4) of double-fault expression (19) (that is, "any point in $FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} = 1$ and $x_{l_1}x_{l_2} = 1$" where x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}).

The SPMFP strategy requires to select test cases from $FP(S)$ to form a set T such that, for every term p_i, every pair of literals $x^i_{j_1}$ and $x^i_{j_2}$ in p_i, and every pair of missing literals x_{l_1} and x_{l_2} of p_i,

1. there is a test case $\vec{t}_1 \in T$ such that, $p_{i,\bar{j}_1,\bar{j}_2} = 1$, $x_{l_1} = 0$ and $x_{l_2} = 0$, if possible;
2. there is a test case $\vec{t}_2 \in T$ such that, $p_{i,\bar{j}_1,\bar{j}_2} = 1$, $x_{l_1} = 0$ and $x_{l_2} = 1$, if possible;
3. there is a test case $\vec{t}_3 \in T$ such that, $p_{i,\bar{j}_1,\bar{j}_2} = 1$, $x_{l_1} = 1$ and $x_{l_2} = 0$, if possible; and
4. there is a test case $\vec{t}_4 \in T$ such that, $p_{i,\bar{j}_1,\bar{j}_2} = 1$, $x_{l_1} = 1$ and $x_{l_2} = 1$, if possible.

In other words, points in T are from $FP(S)$ such that, for every term p_i, (1) $p_{i,\bar{j}_1,\bar{j}_2} = 1$ for every pair of literals $x^i_{j_1}$ and $x^i_{j_2}$ in p_i and (2) they cover all possible truth value combinations (that is `00`, `01`, `10`

Table 17: All false points of S where $S = ab + cd + ef$

000000, 000001, 000010, 000100, 000101, 000110, 001000, 001001, 001010, 010000, 010001, 010010, 010100, 010101, 010110, 011000, 011001, 011010, 100000, 100001, 100010, 100100, 100101, 100110, 101000, 101010, 101001

and 11) of every pair of two different missing literals x_{l_1} and x_{l_2} of p_i. Example 6.7 illustrates how to select test cases satisfying the SPMFP strategy.

Example 6.7 Let $S = ab + cd + ef$. Table 17 lists the set $FP(S)$ of all false points of S. Now, let T = $\{$000000, 000101, 000110, 001001, 001010, 010001, 010010, 010100, 011000, 100001, 100010, 100100, 101000$\}$. Test cases in T are underlined in Table 17 for ease of reference. Test cases in T satisfy the SPMFP strategy because of the following reasons

1. For the first term ab, the test cases 000000, 000101, 000110, 001001 and 001010 in $FP(S)$ are such that $\bar{a}\bar{b} = 1$ and they cover all possible truth value combinations of every possible pair of missing literals (c, d, e and f) of ab. It should be noted that it is impossible to select points from $FP(S)$ such that both c and d evaluate to 1 because such points are true points of $p_2 = cd$ and, hence, are true points of S. Similary, it is impossible to select points from $FP(S)$ such that both e and f evaluate to 1 because such points are true points of $p_3 = ef$. Hence, these 5 test cases satisfy the requirements of the SPMFP strategy for ab of S.

2. Similarly, the test cases 000000, 010001, 010010, 100001 and 100010 in $FP(S)$ are such that $\bar{c}\bar{d} = 1$ and they cover all possible truth value combinations of every possible pair of missing literals (a, b, e and f) of cd. It should be noted that it is impossible to select points from $FP(S)$ such that both a and b evaluate to 1 because such points are true points of $p_1 = ab$ and, hence, are true points of S. Similary, it is impossible to select points from $FP(S)$ such that both e and f evaluate to 1 because such points are true points of $p_3 = ef$. Hence, these 5 test cases satisfy the requirements of the SPMFP strategy for cd of S.

3. Finally, the test cases 000000, 010100, 011000, 100100 and 101000 in $FP(S)$ are such that

$\bar{e}\bar{f} = 1$ and they cover all possible truth value combinations of every possible pair of missing literals (a, b, c and d) of ef. It should be noted that it is impossible to select points from $FP(S)$ such that both a and b evaluate to 1 because such points are true points of $p_1 = ab$ and, hence, are true points of S. Similary, it is impossible to select points from $FP(S)$ such that both c and d evaluate to 1 because such points are true points of $p_2 = cd$. Hence, these 5 test cases satisfy the requirements of the SPMFP strategy for ef of S. $\Diamond$

We now prove that the SPMFP, PMNFP and MUMCUT strategies select test cases to detect double-fault expression (19) in Theorem 6.6.

Theorem 6.6 *Let $S = p_1 + \cdots + p_m$ be a Boolean specification in IDNF. Suppose that the j_1-th and j_2-th literals, $x^{i_1}_{j_1}$ and $x^{i_1}_{j_2}$, of the i_1-th term, p_{i_1}, in S are replaced by x_{l_1} and x_{l_2}, respectively, where $1 \le i_1 \le m$, $1 \le j_1 < j_2 \le k_{i_1}$, k_{i_1} is the number of literals in p_{i_1}, and x_{l_1} and x_{l_2} are two different missing literals of p_{i_1}, the resulting implementation denoted by I will be equivalent to that given by double-fault expression (19) in Table 1 and its detection conditions are given by the corresponding conditions (C1) to (C5) of double-fault expression (19) in Table 2. The SPMFP and PMNFP strategies can supplement the MUMCUT strategy to guarantee the detection of double-fault expression (19) provided that $S \not\equiv I$.*

Proof : Since S and I are not equivalent, by Theorem 4.19, there is a point $\vec{t}_1$ that satisfies any of the following conditions

(C1) $\vec{t}_1 \in UTP_{i_1}(S)$ such that $x_{l_1} = 0$,

(C2) $\vec{t}_1 \in UTP_{i_1}(S)$ such that $x_{l_2} = 0$,

(C3) $\vec{t}_1 \in NFP_{i_1,\bar{j}_1}(S)$ such that $x_{l_1}x_{l_2} = 1$,

(C4) $\vec{t}_1 \in NFP_{i_1,\bar{j}_2}(S)$ such that $x_{l_1}x_{l_2} = 1$, or

(C5) $\vec{t}_1 \in FP(S)$ such that $p_{i_1,\bar{j}_1,\bar{j}_2} = 1$ and $x_{l_1}x_{l_2} = 1$.

We then have the following cases:

Case 1 $\vec{t}_1$ satisfies (C1). Then, $\{\vec{t} \in UTP_{i_1}(S) : x_{l_1}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_1} is a missing literal of p_{i_1}, the MUTP strategy can select a point that satisfies (C1).

Case 2 $\vec{t}_1$ satisfies (C2). Then, $\{\vec{t} \in UTP_{i_1}(S) : x_{l_2}(\vec{t}) = 0\} \neq \emptyset$. Since x_{l_2} is a missing literal of p_{i_1}, the MUTP strategy can select a point that satisfies (C2).

Case 3 $\vec{t}_1$ satisfies (C3). Then, $\{\vec{t} \in NFP_{i_1,\bar{j}_1}(S) : x_{l_1}x_{l_2}(\vec{t}) = 1\} = \{\vec{t} \in NFP_{i_1,\bar{j}_1}(S) : x_{l_1}(\vec{t}) = 1 \textit{ and } x_{l_2}(\vec{t}) = 1\} \neq \emptyset$. The PMNFP strategy can select a point that satisfies (C3).

Case 4 $\vec{t}_1$ satisfies (C4). Then, $\{\vec{t} \in NFP_{i_1,\bar{j}_2}(S) : x_{l_1}x_{l_2}(\vec{t}) = 1\} = \{\vec{t} \in NFP_{i_1,\bar{j}_2}(S) : x_{l_1}(\vec{t}) = 1 \textit{ and } x_{l_2}(\vec{t}) = 1\} \neq \emptyset$. The PMNFP strategy can select a point that satisfies (C4).

Case 5 $\vec{t}_1$ satisfies (C5). Then, $\{\vec{t} \in FP(S) : p_{i,\bar{j}_1,\bar{j}_2}x_{l_1}x_{l_2}(\vec{t}) = 1\} = \{\vec{t} \in FP(S) : p_{i,\bar{j}_1,\bar{j}_2}(\vec{t}) = 1, x_{l_1}(\vec{t}) = 1 \textit{ and } x_{l_2}(\vec{t}) = 1\} \neq \emptyset$. The SPMFP strategy can select a point that satisfies (C5).

Hence, the result follows. □

7 Conclusion and Further Work

In this report, we study the detection conditions on double faults related to literals that might occur in a Boolean expression. As reported in [6], there are 19 such double-fault expressions.

Previous study in double faults related to terms [7] shows that any test case selection strategy which subsumes the BASIC strategy (such as the MUMCUT, MAX-A and MAX-B strategies) can guarantee the detection of double faults related to terms. However, as discussed in Section 5, none of the abovementioned test case selection strategies can guarantee to detect double literal faults studied in this report. Hence, new test case selection strategies are required.

In this report, we propose six new test case selection strategies to supplement the MUMCUT strategy to detect all double literal faults. Hence, when all these strategies are applied, both single and double faults can be detected. They are the Supplementary Multiple False Point (SMFP) strategy, Supplementary Multiple Overlapping True Point (SMOTP) strategy , Supplementary Multiple Unique True Point (SMUTP) strategy, Pairwise Multiple Unique True Point (PMUTP) strategy, Pairwise Multiple Near False Point (PMNFP) strategy and Supplementary Pairwise Multiple False Point (SPMFP) strategy. As a result, the MUMCUT strategy together with these six newly proposed strategies can guarantee to detect all double faults studied in this report.

Empirical study is underway to evaluate the cost-effectiveness of these six newly proposed test case selection strategies together with the MUMCUT strategy for detecting all double literal faults in Boolean expressions. Furthermore, we will continue to study double faults in Boolean expressions in which one fault is related to term and another fault is related to literal. Through our study on the detection of double faults, we will be able to gain better understanding on the characteristics of multiple faults and proposing new strategies to detect them.

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