Faculty of Information and Communication Technologies

Detecting Double Faults Related to Terms in Boolean Expression

Technical Report: SUTICT-TR2006.02

Man Fai Lau and Ying Liu

First version: 30th Nov 2006

Last revision: 17th October 2008 - Fixing minor errors and Redo Tables 2 and 4

for consistency with with other reports

SWINBURNE UNIVERSITY
OF TECHNOLOGY

This page is intentionally left blank.

Detecting Double Faults Related to Terms in Boolean Expression

Table of contents

1 Introduction

Fault-based testing techniques have been proposed aiming at detection of hypothesized faults in a program. More precisely, if an hypothesized fault has been injected in the program, the test cases generated by the fault-based testing techniques will be able to reveal the corresponding failures of the faulty program. As a result, test sets generated aiming at the hypothesized fault will be able to detect the faulty program. Many techniques of fault-based testing have been developed and widely used [2, 4, 11, 12, 17]. Symbolic testing has been proposed as a fault-based testing strategy in [11], and it has been further applied in [2]. Moreover, mutation testing, a typical fault-based testing technique, has been used for many decades [4, 12, 17].

Recently, the detection conditions of hypothesized fault classes have been widely studied [1, 6, 9, 15]. The fault detection conditions of hypothesized faults are mainly used in two areas. First, they are used to develop test case selection strategies aiming at the detection of particular types of faults. Chen and Lau [1] proposed three test case selection strategies based on fault detection conditions of seven types of faults. Second, fault detection conditions are used to develop *fault class hierarchies* [6, 9, 15]. Fault class hierarchy establishes relationships between different types of faults. If any test cases that can be used to detect fault class A can be used to detect fault class B, fault class A is put in the hierarchy lower than that of fault class B. Kuhn [6] and Lau and Yu [9] used fault class hierarchies to explain the empirical results for existing fault-based testing methodologies.

Most previous studies on fault detection conditions assume that only one of these hypothesized faults occur in the software under test. However, investigations show that multiple faults occur more frequently in programs [10, 16]. Research on multiple faults is mainly focus on *fault coupling* of double fault. Double faults is a special instance of multiple faults. *Fault coupling* refers to the situation that two faults can be detected individually, but fail to be detected when they combined together.

Fault coupling have been studied previously [5, 12]. We will only briefly described these previous work here. Detailed discussions can be found in Section 5. Offutt performed an empirical study on fault coupling with 3 small programs via mutation analysis [12]. Given a program, a 1-order (2-order) *mutant* is a program that differs from the original program by 1 syntactic change (2 syntactic changes). In [12], Offutt generated all possible 1-order mutants for the studied programs, discarded all the 1-order mutants that were equivalent to the original program, and generated all 2-order mutant based on those remaining 1-order mutants. Then, he generated test sets that can kill the remaining 1-order mutants and used these test sets to kill 2-order mutants. He found that most test sets that can kill 1-order mutants can also be used to kill a high percentatge of 2-order mutants. However, there is no formal analysis related to high detection rate against the types of mutation operators being used to generate those mutants.

On the other hand, How Tai Wah study the fault coupling via theoretical study [5]. He modeled program as composition of functions while program with single fault (double faults, respectively) is denoted as a composition in which exactly one function is (two functions are, respectively) faulty. Test sets that detect individual faults of a double fault are called *proper test sets*. Among the proper test sets, those that cannot detect the double fault are called *coupled test sets*. He then calculates the *coupling ratio*, defined as the ratio of the number of coupled test sets to that of proper test sets. The mathematical analysis shows that the coupling ratio is very small, therefore How Tai Wah concluded that fault coupling rarely occurs. Since faulty program is modeled by incorrect use of one (respectively, two) of the functions, no fault types have been actually involved in this study.

Recently, Lau and Yu extended their study on using fault class hierarchy to study double faults related to terms [8]. They found that test case that can detect some particular classes of faults in the lower part of the hierarchy can detect double faults which involve these classes of faults and faults in the upper part of the hierarchy. For example, if a test case that can detect fault class A which is lower than fault class B in the fault class hierarchy, then the same test case can detect the double fault involving fault classes A and B, denoted as $A \bowtie B$.

In this report, our main target is to obtain the fault detection conditions of double faults related to terms. In order to illustrate how these fault detection conditions can be used, we analyse the fault detection capabilities of some existing test case selection strategies, aimed at detecting single occurrence of particular types of fault, in detecting double faults studied in this report. Our analysis shows that test case selection strategies studied in this report can guarantee to detect the double faults related to terms. Hence, by using these test case selection strategies in detecting single faults, there is an additional benefit of detecting the double faults studied in this report.

The rest of report is organized as follows. Section 2 introduces the notation and fault classes studied in this report. Section 3 presents double fault classes and their corresponding faulty implementations. Section 4 proves the fault detection conditions of each faulty implementation and Section 5 analyses the effects of fault coupling between single and double faults. Section 6 analyses the fault detecting capabilities of some existing test case selection strategies in detecting double faults. Section 7 concludes the report and discusses further work.

2 Preliminary

2.1 Notation

In this report, we use '$\cdot$', '+' and '$^{-}$' to represent Boolean operators, AND, OR and NOT, respectively. Usually, '$\cdot$' is omitted whenever it is clear from the context. We use 1 and 0 to represent the truth values 'TRUE' and 'FALSE', respectively. The set of all truth values, that is $\{0,1\}$, is denoted as $\mathbb{B}$.

Let S be a Boolean expression in disjunctive normal form

$$S = p_1 + \cdots + p_m$$

where m is the number of terms, $p_i = x^i_1 \cdots x^i_{k_i}$ is the i-th term of S, x^i_j is the j-th literal in p_i, and k_i is the number of literals in p_i. A Boolean expression is in *irredundant disjunctive normal form* (or, simply *IDNF*) if (1) none of its terms can be omitted from the expression; and (2) none of its literals can be omitted from any term in the expression without affecting the function of the expression.

Let S be a Boolean expression having n variables, the input domain is the n-dimensional Boolean space $\mathbb{B}^n$. *True points* of S are those points in $\mathbb{B}^n$ that make S evaluate to 1. The set of all true points of S is denoted by $TP(S)$. A true point of the term p_i in S is a point that makes p_i evaluate to 1. The set of all true points of p_i in S is denoted by $TP_i(S)$. Hence, $TP(S) = \bigcup_i TP_i(S)$. A *unique true point* of p_i in S is a true point of S that makes (1) p_i evaluate to 1; and (2) all other terms evaluate to 0. The set of all unique true points of p_i in S is denoted by $UTP_i(S)$. The set of all unique true points of S is denoted by $UTP(S)$ and $UTP(S) = \bigcup_i UTP_i(S)$.

False points of S are those points in $\mathbb{B}^n$ that make S evaluate to 0 and the set of all false points of S is denoted by $FP(S)$. A *near false point* of the j-th literal, x^i_j, of the i-th term, p_i, in S is a false point that makes (1) x^i_j evaluate to 0, and (2) all other literals in p_i evaluate to 1. The set of all near false points for the j-th literal x^i_j of the i-th term, p_i, in S is denoted by $NFP_{i,\bar{j}}(S)$. The set of all near false points for the i-th term, p_i, in S is denoted by $NFP_i(S)$. Therefore, $NFP_i(S)$= $\bigcup_j NFP_{i,\bar{j}}(S)$. The set of all near false points of S is denoted by $NFP(S)$ and $NFP(S) = \bigcup_i NFP_i(S)$.

2.2 Fault Types

In this report, we only consider five types of faults related to terms in Boolean expressions. Let S be a Boolean expression. Suppose that a single fault F changes a subexpression E into a subexpression E', the resulting faulty expression (or, implementation) is denoted as $I_{F(E \to E')}$. A faulty implementation is referred to as *single-fault expression* if (1) it differs from the original expression by one syntactic change; and (2) it is not equivalent to the original expression. The following five fault types related to terms in a Boolean expression are studied in this report:

1. *Expression Negation Fault* (ENF): The entire Boolean expression or its sub-expression is implemented as its negation. The faulty implementation is $I_{ENF(S \to \overline{S})} = \overline{S}$ or $I_{ENF(p_{i_1}+\cdots+p_{h_1}}$

$\rightarrow\overline{p_{i_1}+\cdots+p_{h_1}})} = p_1 + \cdots + p_{i_1-1} + \overline{p_{i_1} + \cdots + p_{h_1}} + p_{h_1+1} + \cdots + p_m$ where $1 \leq i_1 < h_1 \leq m$. For example, the Boolean expression $ab+cd+ef$ may be implemented as $\overline{ab+cd+ef}$ or $ab+\overline{cd+ef}$. The corresponding detection condition, denoted by DC_{ENF}, is "any point in $\left(\bigcup_{i=i_1}^{h_1} TP_i(S)\right) \setminus \left(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\right)$ or any point in $FP(S)$".

2. *Term Negation Fault* (TNF): A particular term in the Boolean expression is implemented as its negation. The faulty implementation is $I_{TNF(p_{i_1}\rightarrow\overline{p_{i_1}})} = p_1 + \cdots + \overline{p_{i_1}} + \cdots + p_m$, where $1 \leq i_1 \leq m$ and $m > 1$. For example, the Boolean expression $ab+cd+ef$ may be implemented as $ab+cd+\overline{ef}$. The corresponding detection condition, denoted by DC_{TNF}, is "any point in $UTP_{i_1}(S)$ or any point in $FP(S)$". As documented in [9], when S contains just one term (that is m = 1), the negation fault is considered as an ENF.

3. *Term Omission Fault* (TOF): A particular term in the Boolean expression is omitted in its implementation. The faulty implementation is $I_{TOF(p_{i_1}\rightarrow)} = p_1 + \cdots + p_{i_1-1} + p_{i_1+1} + \cdots + p_m$, where $1 \leq i_1 \leq m$ and $m > 1$. For example, the Boolean expression $ab+cd+ef$ may be implemented as $ab+cd$. The corresponding detection condition, denoted by DC_{TOF}, is "any point in $UTP_{i_1}(S)$".

4. *Disjunctive Operator Reference Fault* (DORF): A Boolean operator '$+$' is implemented as '$\cdot$'. The faulty implementation is $I_{DORF(p_{i_1}+p_{i_1+1}\rightarrow p_{i_1}p_{i_1+1})} = p_1 + \cdots + p_{i_1}p_{i_1+1} + \cdots + p_m$, where $1 \leq i_1 < m$. For example, the Boolean expression $ab+cd+ef$ may be implemented as $ab+cdef$. The corresponding detection condition, denoted by DC_{DORF}, is "any point in $UTP_{i_1}(S)$ or any point in $UTP_{i_1+1}(S)$".

5. *Conjunctive Operator Reference Fault* (CORF): A Boolean operator '$\cdot$' is implemented as '$+$'. The faulty implementation is $I_{CORF(p_{i_1}\rightarrow p_{i_1,1,j_1}+p_{i_1,j_1+1,k_{i_1}})} = p_1 + \cdots + p_{i_1-1} + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_m$, where $1 \leq i_1 \leq m$, $m > 1$, $1 \leq j_1 < k_{i_1}$, $p_{i_1,1,j_1} = x_1^{i_1}\cdots x_{j_1}^{i_1}$, $p_{i_1,j_1+1,k_{i_1}} = x_{j+1}^{i_1}\cdots x_{k_{i_1}}^{i_1}$ and $p_{i_1} = p_{i_1,1,j_1}\cdot p_{i_1,j_1+1,k_{i_1}}$. For example, the Boolean expression $ab+cd+ef$ may be implemented as $ab+c+d+ef$. The corresponding detection condition, denoted by DC_{DORF}, is "any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ = 1".

3 Double Fault Models

Similar to previous studies of double fault [5, 12], a *double fault* is defined as the occurrence of two single faults in this report. When a double fault occurs in a Boolean expression, the resulting expression may be equivalent to the original expression or a faulty expression with a single fault. Lau and Liu [7] defined a *double-fault expression* is one which differs from the original expression by two syntactic changes and is equivalent to neither the original expression nor any expression that results from any single fault.

When a double fault occurs, it is possible that the order of occurrence of the two single faults may result in different faulty expressions. Such a double fault is referred to as *double fault with ordering*. However, there is also a chance that the two faults may result in two faulty expressions that are equivalent to each other no matter which fault occurs first. This type of double fault is referred to as *double fault without ordering*.

As reported by Lau and Liu [7], based on the five single fault classes studied in this report, there are 15 double fault classes in double faults without ordering resulting in 27 possible distinct faulty expressions. For the case of double faults with ordering, there are 25 double fault classes resulting in 53 possible faulty expressions. Lau and Liu also found that 49 out of these 53 double-fault expressions have their equivalent double-fault expressions in double faults without ordering. Only 4 double-fault expressions based on double faults with ordering do not have their equivalent counterparts in double faults without ordering. Together with 27 double-fault expressions of double faults without ordering, there are 31 distinct double-fault expressions to be studied in this report. In this section, we introduce all 31 double-fault expressions. The detection conditions of all these double-fault expressions will be studied in Section 4.

For any two single fault classes A and B, we use the notation $A \bowtie B$ to denote the double fault class formed from A and B, that is, the class of faults due to the occurrences of two faults: one fault of class A and another fault of class B. Given a Boolean expression S, suppose two faults

Table 1: Double fault, double-fault expression and detection condition ($S = p_1 + \ldots + p_m$)

(a) Double-fault expressions (1 – 27)[a] due to double fault without ordering

Fault class	Double-fault expression
ENF $\bowtie$ ENF	Case 1 ($i_1 < h_1 < i_2 < h_2$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{h_1}} + \cdots + \overline{p_{i_2} + \cdots + p_{h_2}} + \cdots + p_m$ (1)
	Case 2 ($i_1 \leq i_2 < h_2 \leq h_1$ and $\{i_2, \ldots, h_2\} \subsetneqq \{i_1, \ldots, h_1\}$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + \overline{p_{i_2} + \cdots + \cdots + p_{h_2}} + \cdots + p_{h_1}} + \cdots + p_m$ (2)
ENF $\bowtie$ TNF	Case 1 ($i_1 < h_1 < i_2$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{h_1}} + \cdots + \overline{p_{i_2}} + \cdots + p_m$ (3)
	Case 2 ($i_1 \leq i_2 \leq h_1$ and $i_1 < h_1$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + \overline{p_{i_2}} + \cdots + p_{h_1}} + \cdots + p_m$ (4)
ENF $\bowtie$ TOF	Case 1 ($i_1 < h_1 < i_2$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{h_1}} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_m$ (5)
	Case 2 ($i_1 \leq i_2 \leq h_1$ and $i_1 < h_1$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}} + \cdots + p_m$ (6)
ENF $\bowtie$ DORF	Case 1 ($i_1 < h_1 < i_2$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{h_1}} + \cdots + p_{i_2} p_{i_2+1} + \cdots + p_m$ (7)
	Case 2 ($i_1 < h_1 < m$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{h_1}} p_{h_1+1} + \cdots + p_m$ (8)
	Case 3 ($i_1 < m$): $p_1 + \cdots + \overline{p_{i_1} p_{i_1+1}} + \cdots + p_m$ (9)
	Case 4 ($i_1 \leq i_2 < h_1$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{i_2} p_{i_2+1} + \cdots + p_{h_1}} + \cdots + p_m$ (10)
ENF $\bowtie$ CORF	Case 1 ($i_1 < h_1 < i_2$):$p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{h_1}} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_m$ (11)
	Case 2 ($i_1 \leq i_2 \leq h_1$ and $i_1 < h_1$): $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_{h_1}} + \cdots + p_m$ (12)
TNF $\bowtie$ TNF	($i_1 < i_2$): $p_1 + \cdots + \overline{p_{i_1}} + \cdots + \overline{p_{i_2}} + \cdots + p_m$ (13)
TNF $\bowtie$ TOF	($i_1 < i_2$): $p_1 + \cdots + \overline{p_{i_1}} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_m$ (14)
TNF $\bowtie$ DORF	Case 1 ($i_1 < i_2 < m$): $p_1 + \cdots + \overline{p_{i_1}} + \cdots + p_{i_2} p_{i_2+1} + \cdots + p_m$ (15)
	Case 2 ($i_1 < m$):$p_1 + \cdots + \overline{p_{i_1}} p_{i_1+1} + \cdots + p_m$ (16)
TNF $\bowtie$ CORF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + \overline{p_{i_1}} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_m$ (17)
	Case 2 (both faults occur at p_{i_2}): $p_1 + \cdots + \overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}} + \cdots + p_m$ (18)
TOF $\bowtie$ TOF	($i_1 < i_2$): $p_1 + \cdots + p_{i_1-1} + p_{i_1+1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_m$ (19)
TOF $\bowtie$ DORF	($i_1 < i_2 < m$): $p_1 + \cdots + p_{i_1-1} + p_{i_1+1} + \cdots + p_{i_2} p_{i_2+1} + \cdots + p_m$ (20)
TOF $\bowtie$ CORF	($i_1 < i_2$): $p_1 + \cdots + p_{i_1-1} + p_{i_1+1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_m$ (21)
DORF $\bowtie$ DORF	Case 1 ($i_1 < i_2 < m$): $p_1 + \cdots + p_{i_1} p_{i_1+1} + \cdots + p_{i_2} p_{i_2+1} + \cdots + p_m$ (22)
	Case 2 ($i_1 < m - 1$): $p_1 + \cdots + p_{i_1} p_{i_1+1} p_{i_1+2} + \cdots + p_m$ (23)
DORF $\bowtie$ CORF	Case 1 ($i_1 < i_2 - 1$): $p_1 + \cdots + p_{i_1} p_{i_1+1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_m$ (24)
	Case 2 ($i_1 < m$): $p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} p_{i_1+1} + \cdots + p_m$ (25)
CORF $\bowtie$ CORF	Case 1 ($i_1 < i_2$): $p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_m$ (26)
	Case 2 (both faults occur at p_{i_1}): $p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} + \cdots + p_m$ (27)

(b) Four double-fault expressions (53, 70, 73, and 76)[a] due to double fault with ordering

Fault class	Double-fault expression
TOF $\bowtie$ DORF	($1 < i_1 < m$): $p_1 + \cdots + p_{i_1-1} p_{i_1+1} + \cdots + p_m$ (53)
CORF $\bowtie$ ENF	($i_1 < h_2$): $p_1 + \cdots + p_{i_1,1,j_1} + \overline{p_{i_1,j_1+1,k_{i_1}} + \cdots p_{h_1}} + \cdots + p_m$ (70)
CORF $\bowtie$ TNF	$p_1 + \cdots + p_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}} + \cdots + p_m$ (73)
CORF $\bowtie$ TOF	$p_1 + \cdots + p_{i_1,1,j_1} + \cdots + p_m$ (76)

[a] For ease of cross-reference, the numbering of the faulty expressions follows that of [7].

A and B are committed on the expression changing E_1 and E_2 in the expression to E_1' and E_2', respectively, the resulting faulty expression (or, implementation) is denoted as $I_{A(E_1 \to E_1') \bowtie B(E_2 \to E_2')}$. Table 1 lists all these double fault classes and their corresponding double-fault expressions. We use the same expression numbers of those double-fault expressions reported in [7] for consistency purposes. Let us consider the third row of Table 1, which is concerned with ENF $\bowtie$ TOF. Let S be a Boolean expression in IDNF. There are two subcases. First, the term p_{i_2} is not contained in the subexpression $p_{i_1} + \cdots + p_{h_1}$. Without loss of generality, we may assume that $h_1 < i_2$. The double-fault expression is equivalent to $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{h_1}} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_m$. Second, the term p_{i_2} is contained in the subexpression $p_{i_1} + \cdots + p_{h_1}$, that is, $i_1 \leq i_2 \leq h_1$ and $i_1 < h_1$. The faulty expression is then equivalent to $p_1 + \cdots + \overline{p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}} + \cdots + p_m$.

4 Detection Conditions of Various Double Fault Classes

The detection conditions of hypothesized faults have been studied recently [1, 6, 9, 15]. Let S be a specification and I be the expression which differs from S by several syntactic changes. Whenever S and I evaluate to different values, they can be distinguished from each other. The *detection condition* of I with respect to S is a condition that makes S and I evaluate to different values. As a result, the Boolean exclusive-or operator `XOR`, denoted as $\oplus$, can be used to find the detection conditions. In short, the detection condition can be derived from $S \oplus I$.

In this section, we prove the detection conditions of all 31 double-fault expressions listed in Table 1. The detection conditions of the 27 faulty expresssions of double fault without ordering are proved in Section 4.1 whereas the detection conditions of the remaining 4 faulty expressions are proved in Section 4.2. Instead of simply presenting the Boolean expression $S \oplus I$ as detection conditions, we present them as conditions satisfied by test cases in $\mathbb{B}^n$. Since such categorization is based on certain properties of test sets, it helps in identifying and developing test case selection strategy to detect such double faults in Table 1.

4.1 Detection Conditions on Double Term Faults Without Ordering

4.1.1 ENF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is an ENF.

Theorem 4.1 *(ENF $\bowtie$ ENF - Case 1)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two subexpressions $(p_{i_1} + \cdots + p_{h_1})$ and $(p_{i_2} + \cdots + p_{h_2})$ in S are negated where $1 \le i_1 < h_1 < i_2 < h_2 \le m$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie ENF(p_{i_2}+\cdots+p_{h_2} \to \overline{p_{i_2}+\cdots+p_{h_2}})}$ is equivalent to that given by Expression (1) in Table 1. Then, we have $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie ENF(p_{i_2}+\cdots+p_{h_2} \to \overline{p_{i_2}+\cdots+p_{h_2}})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big) \cap \big(\bigcup_{i=i_2}^{h_2} TP_i(S) \big) \Big) \setminus \big(\bigcup_{i=1, i \ne i_1,\ldots,h_1,i_2,\ldots,h_2}^{m} TP_i(S) \big)$, *or*
2. $\vec{t} \in FP(S)$.

Proof : First, we observe that $S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie ENF(p_{i_2}+\cdots+p_{h_2} \to \overline{p_{i_2}+\cdots+p_{h_2}})}$

$$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2}) \oplus (\overline{p_{i_1} + \cdots + p_{h_1}} + \overline{p_{i_2} + \cdots + p_{h_2}}) \big)$$

$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{h_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2}) \cdot (\overline{\overline{p_{i_1} + \cdots + p_{h_1}} + \overline{p_{i_2} + \cdots + p_{h_2}}})$$

$$+ (\overline{p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2}}) \cdot (\overline{p_{i_1} + \cdots + p_{h_1}} + \overline{p_{i_2} + \cdots + p_{h_2}}) \big)$$

$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{h_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2} + \cdots + p_{h_2}) \cdot (\overline{\overline{p_{i_1} + \cdots + p_{h_1}}} \cdot \overline{\overline{p_{i_2} + \cdots + p_{h_2}}})$$

$$+ (\overline{p_{i_1} + \cdots + p_{h_1}} \cdot \overline{p_{i_2} + \cdots + p_{h_2}}) \cdot (\overline{p_{i_1} + \cdots + p_{h_1}} + \overline{p_{i_2} + \cdots + p_{h_2}}) \big)$$

$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{h_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{i_2}+\cdots+p_{h_2})\cdot(p_{i_1}+\cdots+p_{h_1})\cdot(p_{i_2}+\cdots+p_{h_2})$$
$$+(\overline{p_{i_1}+\cdots+p_{h_1}}\cdot\overline{p_{i_2}+\cdots+p_{h_2}})\cdot(\overline{p_{i_1}+\cdots+p_{h_1}}+\overline{p_{i_2}+\cdots+p_{h_2}})\big)$$
$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{h_2+1}\cdots\bar{p}_m$$
$$\equiv \big((p_{i_1}+\cdots+p_{h_1})(p_{i_2}+\cdots+p_{h_2})+(\overline{p_{i_1}+\cdots+p_{h_1}}\cdot\overline{p_{i_2}+\cdots+p_{h_2}})\big)$$
$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{h_2+1}\cdots\bar{p}_m$$
$$\equiv (p_{i_1}+\cdots+p_{h_1})(p_{i_2}+\cdots+p_{h_2})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{h_2+1}\cdots\bar{p}_m$$
$$+(\overline{p_{i_1}+\cdots+p_{h_1}}\cdot\overline{p_{i_2}+\cdots+p_{h_2}})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{h_2+1}\cdots\bar{p}_m$$
$$\equiv (p_{i_1}+\cdots+p_{h_1})(p_{i_2}+\cdots+p_{h_2})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{h_2+1}\cdots\bar{p}_m+\overline{S}$$

Now, $S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1}\rightarrow\overline{p_{i_1}+\cdots+p_{h_1}})\bowtie ENF(p_{i_2}+\cdots+p_{h_2}\rightarrow\overline{p_{i_2}+\cdots+p_{h_2}})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1}\rightarrow\overline{p_{i_1}+\cdots+p_{h_1}})\bowtie ENF(p_{i_2}+\cdots+p_{h_2}\rightarrow\overline{p_{i_2}+\cdots+p_{h_2}})}(\vec{t}) = 1$

if and only if $(p_{i_1}+\cdots+p_{h_1})(p_{i_2}+\cdots+p_{h_2})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{h_2+1}\cdots\bar{p}_m+\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions

1. $\vec{t} \in \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big)\cap\big(\bigcup_{i=i_2}^{h_2} TP_i(S)\big)\Big)\setminus\big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2,\ldots,h_2}}^{m} TP_i(S)\big)$, or
2. $\vec{t} \in FP(S)$.

Hence, the result follows. □

Theorem 4.2 *(ENF ⋈ ENF - Case 2)*

Let $S = p_1+\cdots+p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two subexpressions $(p_{i_1}+\cdots+p_{h_1})$ and $(p_{i_2}+\cdots+p_{h_2})$ in S are negated where $1 \le i_1 \le i_2 < h_2 \le h_1 \le m$ and $\{i_2,\ldots,h_2\} \subsetneqq \{i_1,\ldots,h_1\}$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1}\rightarrow\overline{p_{i_1}+\cdots+p_{h_1}})\bowtie ENF(p_{i_2}+\cdots+p_{h_2}\rightarrow\overline{p_{i_2}+\cdots+p_{h_2}})}$ is equivalent to that given by Expression (2) in Table 1. Then, $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1}\rightarrow\overline{p_{i_1}+\cdots+p_{h_1}})\bowtie ENF(p_{i_2}+\cdots+p_{h_2}\rightarrow\overline{p_{i_2}+\cdots+p_{h_2}})}$ if and only if there is a test case $\vec{t} \in \big(\bigcup_{\substack{i=i_1\\ i\neq i_2,\ldots,h_2}}^{h_1} TP_i(S)\big)\setminus\big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$.

Proof : First, we observe that $S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie ENF(p_{i_2}+\cdots+p_{h_2} \to \overline{p_{i_2}+\cdots+p_{h_2}})}$

$$\equiv \big((p_{i_1}+\cdots+p_{i_2}+\cdots+p_{h_2}+\cdots+p_{h_1}) \oplus (\overline{p_{i_1}+\cdots+\overline{p_{i_2}+\cdots+p_{h_2}}+\cdots+p_{h_1}})\big)$$

$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1})(\overline{\overline{p_{i_1}+\cdots+\overline{p_{i_2}+\cdots+p_{h_2}}+\cdots+p_{h_1}}})$$

$$+(\overline{p_{i_1}+\cdots+p_{h_1}})(\overline{p_{i_1}+\cdots+\overline{p_{i_2}+\cdots+p_{h_2}}+\cdots+p_{h_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1})(p_{i_1}+\cdots+\overline{p_{i_2}+\cdots+p_{h_2}}+\cdots+p_{h_1})$$

$$+(\overline{p_{i_1}+\cdots+p_{h_1}})(\overline{p_{i_1}+\cdots+\overline{p_{i_2}+\cdots+p_{h_2}}+\cdots+p_{h_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{i_2-1}+p_{h_2+1}+\cdots+p_{h_1})+0\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$

(By making use of $(A+B)(A+\overline{B}) \equiv A$ and $(\overline{A+B})(\overline{A+\overline{B}}) \equiv 0$)

$$\equiv (p_{i_1}+\cdots+p_{i_2-1}+p_{h_2+1}+\cdots+p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$

Now, $S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie ENF(p_{i_2}+\cdots+p_{h_2} \to \overline{p_{i_2}+\cdots+p_{h_2}})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie ENF(p_{i_2}+\cdots+p_{h_2} \to \overline{p_{i_2}+\cdots+p_{h_2}})}(\vec{t}) = 1$

if and only if $(p_{i_1}+\cdots+p_{i_2-1}+p_{h_2+1}+\cdots+p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in \Big(\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S) \Big) \setminus \Big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S) \Big)$.

Hence, the result follows. □

As a reminder, Theorem 4.2 excludes the case when S contains just 1 term or 2 terms. When S contains just 1 term (that is $S = p_1$), negating it twice is equivalent to S. When S contains just 2 terms (that is $S = p_1 + p_2$), both ENFs require that $i_1 < h_1$ and $i_2 < h_2$. The net result is to negate S twice, which is then equivalent to S.

Theorem 4.3 *(ENF ⋈ TNF - Case 1)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + \cdots + p_{h_1})$ and the i_2-th term, p_{i_2}, in S are negated where $1 \le i_1 < h_1 < i_2 \le m$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}$ is equivalent to that given by Expression (3) in Table 1. Then, $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}$ if *and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:*

1. $\vec{t} \in \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big) \cap \big(TP_{i_2}(S) \big) \Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, \ldots, h_1, i_2}}^{m} TP_i(S) \big)$, *or*

2. $\vec{t} \in FP(S)$.

Proof : First, we observe that $S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}$

$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \oplus (\overline{p_{i_1} + \cdots + p_{h_1}} + \bar{p}_{i_2}) \big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (\overline{\overline{p_{i_1} + \cdots + p_{h_1}} + \bar{p}_{i_2}}) + (\overline{p_{i_1} + \cdots + p_{h_1} + p_{i_2}}) \cdot (\overline{p_{i_1} + \cdots + p_{h_1}} + \bar{p}_{i_2}) \big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (\overline{\overline{p_{i_1} + \cdots + p_{h_1}}}) \cdot \bar{\bar{p}}_{i_2} + (\overline{p_{i_1} + \cdots + p_{h_1}}) \cdot \bar{p}_{i_2} \cdot (\overline{p_{i_1} + \cdots + p_{h_1}} + \bar{p}_{i_2}) \big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (p_{i_1} + \cdots + p_{h_1}) \cdot p_{i_2} + (\overline{p_{i_1} + \cdots + p_{h_1}}) \cdot \bar{p}_{i_2} \cdot (\overline{p_{i_1} + \cdots + p_{h_1}} + \bar{p}_{i_2}) \big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{h_1}) \cdot p_{i_2} + (\overline{p_{i_1} + \cdots + p_{h_1}}) \cdot \bar{p}_{i_2} \big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv (p_{i_1} + \cdots + p_{h_1}) p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \bar{p}_{i_2-1} \cdots \bar{p}_m$

$+ (\overline{p_{i_1} + \cdots + p_{h_1}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2} \cdots \bar{p}_m$

$\equiv (p_{i_1} + \cdots + p_{h_1}) p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$

Now, $S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}(\vec{t}) = 1$

if and only if $(p_{i_1} + \cdots + p_{h_1}) p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big) \cap \big(TP_{i_2}(S) \big) \Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, \ldots, h_1, i_2}}^{m} TP_i(S) \big)$, or

2. $\vec{t} \in FP(S)$.

Hence, the result follows. $\square$

Theorem 4.4 *(ENF $\bowtie$ TNF - Case 2)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + \cdots + p_{h_1})$ and the i_2-th term, p_{i_2}, in S are negated where $1 \le i_1 \le i_2 \le h_1 \le m$ and $i_1 \ne h_1$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}$ is equivalent to Expression (4) in Table 1. Then, $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}$ if and only if there is a test case $\vec{t} \in \big(\bigcup_{\substack{i=i_1 \\ i \ne i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \ne i_1,\ldots,h_1}}^{m} TP_i(S)\big)$.

Proof : First, we observe that $S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}$

$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \oplus (\overline{p_{i_1} + \cdots + \bar{p}_{i_2} + \cdots + p_{h_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot (\overline{\overline{p_{i_1} + \cdots + \bar{p}_{i_2} + \cdots + p_{h_1}}}) + (\overline{p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}})$

$\cdot (\overline{p_{i_1} + \cdots + \bar{p}_{i_2} + \cdots + p_{h_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot (p_{i_1} + \cdots + \bar{p}_{i_2} + \cdots + p_{h_1}) + (\overline{p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}})$

$\cdot (\overline{p_{i_1} + \cdots + \bar{p}_{i_2} + \cdots + p_{h_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

$\equiv (p_{i_1} + \cdots + p_{i_2-1} + p_{i_{2+1}} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

(By making use of $(A+B)(A+\overline{B}) \equiv A$ and $(\overline{A+B})(\overline{A+\overline{B}}) \equiv 0$)

Now, $\quad S(\vec{t}) \ne I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}(\vec{t})$

if and only if $\quad S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TNF(p_{i_2} \to \bar{p}_{i_2})}(\vec{t}) = 1$

if and only if $\quad (p_{i_1} + \cdots + p_{i_2-1} + p_{i_{2+1}} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\quad \vec{t} \in \big(\bigcup_{\substack{i=i_1 \\ i \ne i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \ne i_1,\ldots,h_1}}^{m} TP_i(S)\big)$.

Hence, the result follows. $\square$

Theorem 4.5 *(ENF $\bowtie$ TOF - Case 1)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose *that the subexpression $(p_{i_1}+\cdots+p_{h_1})$ in S is negated and the i_2-th term, p_{i_2}, in S is omitted where $1 \le i_1 < h_1 < i_2 \le m$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \ltimes TOF(p_{i_2}\to\ )}$ is equivalent to Expression (5) in Table 1. Then, $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \ltimes TOF(p_{i_2}\to\ )}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:*

1. $\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)\big)$, *or*
2. $\vec{t} \in FP(S)$.

Proof : First, we observe that $S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \ltimes TOF(p_{i_2}\to\ )}$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{i_2}) \oplus (\overline{p_{i_1}+\cdots+p_{h_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{i_2}) \cdot (\overline{\overline{p_{i_1}+\cdots+p_{h_1}}}) + (\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}}) \cdot (\overline{p_{i_1}+\cdots+p_{h_1}})\big)$$

$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{i_2})(p_{i_1}+\cdots+p_{h_1}) + (\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}}) \cdot (\overline{p_{i_1}+\cdots+p_{h_1}})\big)$$

$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1}) + (\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv (p_{i_1}+\cdots+p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv (p_{i_1}+\cdots+p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$$

Now, $S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \ltimes TOF(p_{i_2}\to\ )}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \ltimes TOF(p_{i_2}\to\ )}(\vec{t}) = 1$

if and only if $(p_{i_1}+\cdots+p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)\big)$, or
2. $\vec{t} \in FP(S)$.

Hence, the result follows. $\square$

Theorem 4.6 *(ENF ⋈ TOF - Case 2)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + \cdots + p_{h_1})$ in S is negated and the i_2-th term, p_{i_2}, in S is omitted where $1 \leq i_1 \leq i_2 \leq h_1 \leq m$ and $i_1 \neq h_1$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TOF(p_{i_2}\to\ )}$ is equivalent to that given by Expression (6) in Table 1. Then, $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TOF(p_{i_2}\to\ )}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$, *or*
2. $\vec{t} \in FP(S)$.

Proof : First, we observe that

$$S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TOF(p_{i_2}\to\ )}$$
$$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \oplus (\overline{p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}})\big)$$
$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot (\overline{\overline{p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}}}) + (\overline{p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}})$$
$$\cdot (\overline{p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) + (\overline{p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}})$$
$$\cdot (\overline{p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) + (\overline{p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$\equiv (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$+ (\overline{p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$\equiv (p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \overline{S}$$

Now, $S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TOF(p_{i_2} \to\ )}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie TOF(p_{i_2} \to\ )}(\vec{t}) = 1$

if and only if $(p_{i_1} + \cdots + p_{i_2-1} + p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$, or
2. $\vec{t} \in FP(S)$.

Hence, the result follows. □

Theorem 4.7 *(ENF ⋈ DORF - Case 1)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + \cdots + p_{h_1})$ in S is negated and the subexpression $(p_{i_2} + p_{i_2+1})$ in S is implemented as $p_{i_2}p_{i_2+1}$ where $1 \leq i_1 < h_1 < i_2 < m$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ is equivalent to that given in Expression (7) in Table 1. Then, $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$, or*
2. *$\vec{t} \in FP(S)$.*

Proof : First, we observe that

$S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$

$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2} + p_{i_2+1}) \oplus (\overline{p_{i_1} + \cdots + p_{h_1}} + p_{i_2}p_{i_2+1})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{i_2}+p_{i_2+1})\cdot(\overline{\overline{p_{i_1}+\cdots+p_{h_1}}+p_{i_2}p_{i_2+1}})+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}+p_{i_2+1}})$$

$$\cdot(\overline{p_{i_1}+\cdots+p_{h_1}}+p_{i_2}p_{i_2+1})\big)\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{i_2}+p_{i_2+1})\cdot(\overline{\overline{p_{i_1}+\cdots+p_{h_1}}}\cdot\overline{p_{i_2}p_{i_2+1}})+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}+p_{i_2+1}})$$

$$\cdot(\overline{p_{i_1}+\cdots+p_{h_1}}+p_{i_2}p_{i_2+1})\big)\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{i_2}+p_{i_2+1})\cdot((p_{i_1}+\cdots+p_{h_1})\cdot\overline{p_{i_2}p_{i_2+1}})+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}+p_{i_2+1}})$$

$$\cdot(\overline{p_{i_1}+\cdots+p_{h_1}}+p_{i_2}p_{i_2+1})\big)\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1})\overline{p_{i_2}p_{i_2+1}}+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}+p_{i_2+1}})\big)$$

$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

(By making use of $(\overline{A+B+C})(\overline{A}+BC)\equiv\overline{A+B+C}$)

$$\equiv (p_{i_1}+\cdots+p_{h_1})\overline{p_{i_2}p_{i_2+1}}\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}+p_{i_2+1}})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$\equiv (p_{i_1}+\cdots+p_{h_1})\overline{p_{i_2}p_{i_2+1}}\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m+\overline{S}$$

Now, $S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1}\to\overline{p_{i_1}+\cdots+p_{h_1}})\bowtie DORF(p_{i_2}+p_{i_2+1}\to p_{i_2}p_{i_2+1})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1}\to\overline{p_{i_1}+\cdots+p_{h_1}})\bowtie DORF(p_{i_2}+p_{i_2+1}\to p_{i_2}p_{i_2+1})}(\vec{t}) = 1$

if and only if $(p_{i_1}+\cdots+p_{h_1})\overline{p_{i_2}p_{i_2+1}}\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m+\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t}\in\big(\bigcup_{i=i_1}^{h_1}TP_i(S)\big)\setminus\Big(\big(\bigcup_{i=1,\ i\neq i_1,\ldots,h_1,i_2,i_2+1}^{m}TP_i(S)\big)\cup\big(TP_{i_2}(S)\cap TP_{i_2+1}(S)\big)\Big)$,

   or

2. $\vec{t}\in FP(S)$.

Hence, the result follows. □

Theorem 4.8 *(ENF ⋈ DORF - Case 2)*

Let S=$p_1+\cdots+p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1}+\cdots+p_{h_1})$ in S is negated and the subexpression $(p_{i_2}+p_{i_2+1})$ in S

is implemented as $p_{i_2}p_{i_2+1}$ where $1 \leq i_1 < (h_1 = i_2) < m$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ is equivalent to that given by Expression (8) in Table 1. Then, $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ if and only if there is a test case $\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1+1}}^{m} TP_i(S)\big)$.

Proof : Since $i_2 = h_1$, we observe that $S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$

$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{h_1+1}) \oplus (\overline{p_{i_1}+\cdots+p_{h_1}}\,p_{h_1+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{h_1+2} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{h_1+1}) \cdot (\overline{\overline{p_{i_1}+\cdots+p_{h_1}}\,p_{h_1+1}}) + (\overline{p_{i_1}+\cdots+p_{h_1}+p_{h_1+1}})$$
$$\cdot (\overline{p_{i_1}+\cdots+p_{h_1}}\,p_{h_1+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+2} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{h_1+1}) \cdot (\overline{\overline{p_{i_1}+\cdots+p_{h_1}}} + \overline{p_{h_1+1}}) + (\overline{p_{i_1}+\cdots+p_{h_1}+p_{h_1+1}})$$
$$\cdot (\overline{p_{i_1}+\cdots+p_{h_1}}\,p_{h_1+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+2} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1}+\cdots+p_{h_1}+p_{h_1+1}) \cdot (p_{i_1}+\cdots+p_{h_1}+\overline{p_{h_1+1}}) + (\overline{p_{i_1}+\cdots+p_{h_1}} \cdot \bar{p}_{h_1+1})$$
$$\cdot (\overline{p_{i_1}+\cdots+p_{h_1}}\,p_{h_1+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+2} \cdots \bar{p}_m$$
$$\equiv (p_{i_1}+\cdots+p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+2} \cdots \bar{p}_m$$

(By making use of $(A+B)(A+\overline{B}) \equiv A$ and $(A\overline{B})(AB) \equiv 0$)

Now,	$S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t}) = 1$
if and only if	$(p_{i_1}+\cdots+p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+2} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1+1}}^{m} TP_i(S)\big)$.

Hence, the result follows. □

Theorem 4.9 *(ENF ⋈ DORF - Case 3)*

Let S=$p_1+\cdots+p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1}+\cdots+p_{h_1})$ in S is negated and the subexpression $(p_{i_2}+p_{i_2+1})$ in S is implemented as $p_{i_2}p_{i_2+1}$ where $1 \leq (i_1 = i_2) < (h_1 = i_2+1) \leq m$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ is equivalent to that given by Expression (9) *in Table 1. Then,* $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ *if and only if there is a test case* $\vec{t}$ *that satisfies any of the following conditions:*

1. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)$*, or*
2. $\vec{t} \in FP(S)$*.*

Proof : Since $i_1 = i_2$ and $h_1 = i_2 + 1$, we have $h_1 = i_1 + 1$. Then, we observe that

$$S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$$
$$\equiv \big((p_{i_1} + p_{i_1+1}) \oplus (\overline{p_{i_1}p_{i_1+1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1} + p_{i_1+1}) \cdot (\overline{\overline{p_{i_1}p_{i_1+1}}}) + (\overline{p_{i_1} + p_{i_1+1}}) \cdot (\overline{p_{i_1}p_{i_1+1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1} + p_{i_1+1}) \cdot (p_{i_1}p_{i_1+1}) + (\overline{p_{i_1} + p_{i_1+1}}) \cdot (\overline{p_{i_1}p_{i_1+1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$$
$$\equiv (p_{i_1}p_{i_1+1} + (\overline{p_{i_1} + p_{i_1+1}})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$$

(By making use of $(A+B)(AB) \equiv AB$ and $(\overline{A+B})(\overline{AB}) \equiv \overline{A+B}$)

$$\equiv p_{i_1}p_{i_1+1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + (\overline{p_{i_1} + p_{i_1+1}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$$
$$\equiv p_{i_1}p_{i_1+1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + \overline{S}$$

Now, $S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t}) = 1$

if and only if $p_{i_1}p_{i_1+1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)$, or
2. $\vec{t} \in FP(S)$.

Hence, the result follows. □

Theorem 4.10 *(ENF ⋈ DORF - Case 4)*

Let S*=*$p_1 + \cdots + p_m$ *be a Boolean specification in irredundant disjunctive normal form. Suppose* *that the subexpression $(p_{i_1} + \cdots + p_{h_1})$ in S is negated and the subexpression $(p_{i_2} + p_{i_2+1})$ in S is implemented as $p_{i_2}p_{i_2+1}$ where $1 \le i_1 \le i_2 < h_1 \le m$, $i_1 \ne i_2$ and $h_1 \ne i_2+1$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ is equivalent to that given by Expression (10) in Table 1. Then, $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:*

1. $\vec{t} \in \Big(\big(\bigcup_{\substack{i=i_1 \\ i \ne i_2, i_2+1}}^{h_1} TP_i(S) \big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S) \big) \Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \ne i_1,\ldots,h_1}}^{m} TP_i(S) \big)$, *or*
2. $\vec{t} \in FP(S)$.

Proof : First, we observe that $S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$

$\equiv \big((p_{i_1} + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}) \oplus (\overline{p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}}) \big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}) \cdot (\overline{\overline{p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}}})$

$+ (\overline{p_{i_1} + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}}) \cdot (\overline{p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}}) \big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}) \cdot (p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1})$

$+ (\overline{p_{i_1} + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}}) \cdot (\overline{p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}}) \big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) + (\overline{p_{i_1} + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}}) \big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

(By making use of $(A+B+C)(AB+C) \equiv AB+C$ and $(\overline{A+B+C})(\overline{AB+C}) \equiv \overline{A+B+C}$)

$\equiv (p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

$+ (\overline{p_{i_1} + \cdots + p_{i_2} + p_{i_2+1} + \cdots + p_{h_1}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

$\equiv (p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \overline{S}$

Now, $S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t}) = 1$

if and only if $(p_{i_1} + \cdots + p_{i_2}p_{i_2+1} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in \Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2, i_2+1}}^{h_1} TP_i(S) \big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S) \big) \Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S) \big)$, or
2. $\vec{t} \in FP(S)$.

Hence, the result follows. □

Theorem 4.11 *(ENF ⋈ CORF - Case 1)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + \cdots + p_{h_1})$ in S is negated and the i_2-th term, p_{i_2}, in S is implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $1 \leq i_1 < h_1 < i_2 \leq m$, $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$ and $1 \leq j_2 < k_{i_2}$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ is equivalent to Expression (11) in Table 1. Then, $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S) \big)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$, or*
2. *$\vec{t} \in FP(S)$.*

Proof : First, we observe that $S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$

$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \oplus (\overline{p_{i_1} + \cdots + p_{h_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + \cdots + p_{h_1} + p_{i_2}) \cdot (\overline{\overline{p_{i_1} + \cdots + p_{h_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) + (\overline{p_{i_1} + \cdots + p_{h_1} + p_{i_2}})$

$\cdot(\overline{p_{i_1}+\cdots+p_{h_1}}+p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})\big)\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv\big((p_{i_1}+\cdots+p_{h_1}+p_{i_2})\cdot(p_{i_1}+\cdots+p_{h_1})\cdot(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}})+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}})$

$\cdot(\overline{p_{i_1}+\cdots+p_{h_1}}+p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})\big)\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv\big((p_{i_1}+\cdots+p_{h_1})(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}})+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}})\big)$

$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+1}\cdots\bar{p}_m$

(By making use of $(A+B)A\equiv A$ and $(\overline{A+BC})(\overline{A}+B+C)\equiv\overline{A+BC}$)

$\equiv\big((p_{i_1}+\cdots+p_{h_1})(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}})\bar{p}_{i_2}+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}})\big)$

$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+1}\cdots\bar{p}_m$

(By rewriting $\overline{A+B}$ as $(\overline{A+B})(\overline{AB})$ because they are equivalent)

$\equiv(p_{i_1}+\cdots+p_{h_1})(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\bar{p}_{i_2}\bar{p}_{i_2+1}\cdots\bar{p}_m$

$+(\overline{p_{i_1}+\cdots+p_{h_1}+p_{i_2}})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+1}\cdots\bar{p}_m$

$\equiv(p_{i_1}+\cdots+p_{h_1})(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_m+\overline{S}$

Now, $S(\vec{t})\neq I_{ENF(p_{i_1}+\cdots+p_{h_1}\to\overline{p_{i_1}+\cdots+p_{h_1}})\bowtie CORF(p_{i_2}\to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}(\vec{t})$

if and only if $S(\vec{t})\oplus I_{ENF(p_{i_1}+\cdots+p_{h_1}\to\overline{p_{i_1}+\cdots+p_{h_1}})\bowtie CORF(p_{i_2}\to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}(\vec{t})$ = 1

if and only if $(p_{i_1}+\cdots+p_{h_1})(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{h_1+1}\cdots\bar{p}_m+\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t}\in\big(\bigcup_{i=i_1}^{h_1}TP_i(S)\big)\setminus\big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m}TP_i(S)\big)$ such that $p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}$ = 0, or

2. $\vec{t}\in FP(S)$.

Hence, the result follows. □

Theorem 4.12 *(ENF ⋈ CORF - Case 2)*

Let $S=p_1+\cdots+p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1}+\cdots+p_{h_1})$ in S is negated and the i_2-th term, p_{i_2}, in S is implemented *as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $1 \le i_1 \le i_2 \le h_1 \le m$, $i_1 \ne h_1$, $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$ and $1 \le j_2 < k_{i_2}$, the resulting expression denoted as $I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ is equivalent to that given by Expression (12) in Table 1. Then, we have $S \not\equiv I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:*

1. $\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \ne i_1,\ldots,h_1}}^{m} TP_i(S)\big)$*, or*
2. $\vec{t} \in FP(S)$ *such that* $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ *on* $\vec{t}$*.*

Proof : First, we observe that $S \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$

$$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \oplus (\overline{p_{i_1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_{h_1}})\big)$$
$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot (\overline{\overline{p_{i_1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_{h_1}}})$$
$$+ (\overline{p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}}) \cdot (\overline{p_{i_1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_{h_1}})\big)$$
$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot (p_{i_1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_{h_1})$$
$$+ (\overline{p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}}) \cdot (\overline{p_{i_1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_{h_1}})\big)$$
$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) + (\overline{p_{i_1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_{h_1}})\big)$$
$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$

(By making use of $(A+BC)(A+B+C) \equiv A+BC$
and $(\overline{A+BC})(\overline{A+B+C}) \equiv \overline{A+B+C}$)

$$\equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$+ \overline{p_{i_1} + \cdots + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} + \cdots + p_{h_1}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$\equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$
$$+ \bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$
$$\equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$+\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2,1,j_2} + \bar{p}_{i_2,j_2+1,k_{i_2}}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By rewriting AB as $AB(A+B)$ because they are equivalent)

$\equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$

$+\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot (\overline{p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_2} \cdots \bar{p}_m$

$\equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \overline{S}$

$\equiv (p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + (\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) \cdot \overline{S}$

Now, $S(\vec{t}) \neq I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{ENF(p_{i_1}+\cdots+p_{h_1} \to \overline{p_{i_1}+\cdots+p_{h_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}(\vec{t}) = 1$

if and only if $(p_{i_1} + \cdots + p_{i_2} + \cdots + p_{h_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + (\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) \cdot \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$, or
2. $\vec{t} \in FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ on $\vec{t}$.

Hence, the result follows. □

4.1.2 TNF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is a TNF. Since ENF ⋈ TNF has been discussed in previous section, we do not repeat the discussion here.

Theorem 4.13 *(TNF ⋈ TNF)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that *two different terms, p_{i_1} and p_{i_2}, in S are negated where $1 \le i_1 < i_2 \le m$, the resulting expression denoted as $I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \ltimes TNF(p_{i_2} \to \bar{p}_{i_2})}$ is equivalent to that given by Expression (13) in Table 1. Then, $S \not\equiv I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \ltimes TNF(p_{i_2} \to \bar{p}_{i_2})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:*

1. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$, *or*
2. $\vec{t} \in FP(S)$.

Proof : First, we observe that $S \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \ltimes TNF(p_{i_2} \to \bar{p}_{i_2})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (\bar{p}_{i_1} + \bar{p}_{i_2})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{\bar{p}_{i_1} + \bar{p}_{i_2}}) + (\overline{p_{i_1} + p_{i_2}})(\bar{p}_{i_1} + \bar{p}_{i_2})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(p_{i_1} p_{i_2}) + (\overline{p_{i_1} + p_{i_2}})(\bar{p}_{i_1} + \bar{p}_{i_2})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_1} p_{i_2} + (\overline{p_{i_1} + p_{i_2}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $(A+B)(AB) \equiv AB$ and $(\overline{A+B})(\overline{AB}) \equiv \overline{A+B}$)

$\equiv p_{i_1} p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (\overline{p_{i_1} + p_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}$
$\cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_1} p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$

Now, $S(\vec{t}) \neq I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \ltimes TNF(p_{i_2} \to \bar{p}_{i_2})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \ltimes TNF(p_{i_2} \to \bar{p}_{i_2})}(\vec{t})$ = 1

if and only if $p_{i_1} p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in \big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$, or
2. $\vec{t} \in FP(S)$.

Hence, the result follows. □

Theorem 4.14 *(TNF $\bowtie$ TOF)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the i_1-th term, p_{i_1}, in S is negated and the i_2-th term, p_{i_2}, in S is omitted where $1 \leq i_1 < i_2 \leq m$, the resulting expression denoted as $I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie TOF(p_{i_2} \to \ )}$ is equivalent to that given by Expression (14) in Table 1. Then, $S \not\equiv I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie TOF(p_{i_2} \to \ )}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in \big(TP_{i_1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$, *or*

2. $\vec{t} \in FP(S)$.

Proof : First, we observe that $S \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie TOF(p_{i_2} \to \ )}$

$$\equiv \big((p_{i_1} + p_{i_2}) \oplus (\bar{p}_{i_1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2}) \cdot (\bar{\bar{p}}_{i_1}) + (\overline{p_{i_1} + p_{i_2}}) \cdot (\bar{p}_{i_1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2}) \cdot p_{i_1} + (\overline{p_{i_1} + p_{i_2}}) \cdot (\bar{p}_{i_1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big(p_{i_1} + (\overline{p_{i_1} + p_{i_2}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv p_{i_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (\overline{p_{i_1} + p_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv p_{i_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$$

Now,	$S(\vec{t}) \neq I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie TOF(p_{i_2} \to \ )}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie TOF(p_{i_2} \to \ )}(\vec{t}) = 1$
if and only if	$p_{i_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in \big(TP_{i_1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$, or

2. $\vec{t} \in FP(S)$.

Hence, the result follows. □

Theorem 4.15 *(TNF ⋊ DORF - Case 1)*

Let S=$p_1+\cdots+p_m$ *be a Boolean specification in irredundant disjunctive normal form. Suppose that the* i_1*-th term,* p_{i_1}*, in* S *is negated and the subexpression* $(p_{i_2}+p_{i_2+1})$ *in* S *is implemented as* $p_{i_2}p_{i_2+1}$ *where* $1 \leq i_1 < i_2 < m$*, the resulting expression denoted as* $I_{TNF(p_{i_1}\to\bar{p}_{i_1})\rtimes DORF(p_{i_2}+p_{i_2+1}\to p_{i_2}p_{i_2+1})}$ *is equivalent to that given by Expression (15) in Table 1. Then,* $S \not\equiv I_{TNF(p_{i_1}\to\bar{p}_{i_1})\rtimes DORF(p_{i_2}+p_{i_2+1}\to p_{i_2}p_{i_2+1})}$ *if and only if there is a test case* $\vec{t}$ *that satisfies any of the following conditions:*

1. $\vec{t} \in TP_{i_1}(S) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i\neq i_1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$, *or*
2. $\vec{t} \in FP(S)$.

Proof : First, we observe that $S \oplus I_{TNF(p_{i_1}\to\bar{p}_{i_1})\rtimes DORF(p_{i_2}+p_{i_2+1}\to p_{i_2}p_{i_2+1})}$

$$\equiv \big((p_{i_1}+p_{i_2}+p_{i_2+1}) \oplus (\bar{p}_{i_1}+p_{i_2}p_{i_2+1})\big)\cdot \bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$\equiv \big((p_{i_1}+p_{i_2}+p_{i_2+1})\cdot(\overline{\bar{p}_{i_1}+p_{i_2}p_{i_2+1}}) + (\overline{p_{i_1}+p_{i_2}+p_{i_2+1}})\cdot(\bar{p}_{i_1}+p_{i_2}p_{i_2+1})\big)$$

$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$\equiv \big((p_{i_1}+p_{i_2}+p_{i_2+1})\cdot(\bar{\bar{p}}_{i_1}\cdot\overline{p_{i_2}p_{i_2+1}}) + (\overline{p_{i_1}+p_{i_2}+p_{i_2+1}})\cdot(\bar{p}_{i_1}+p_{i_2}p_{i_2+1})\big)$$

$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$\equiv \big((p_{i_1}+p_{i_2}+p_{i_2+1})\cdot(p_{i_1}\cdot\overline{p_{i_2}p_{i_2+1}}) + (\overline{p_{i_1}+p_{i_2}+p_{i_2+1}})\cdot(\bar{p}_{i_1}+p_{i_2}p_{i_2+1})\big)$$

$$\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$\equiv \big(p_{i_1}(\overline{p_{i_2}p_{i_2+1}}) + \overline{p_{i_1}+p_{i_2}+p_{i_2+1}}\big)\ \cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

(By making use of $(\overline{A+B+C})(\overline{A}+BC) \equiv \overline{A+B+C}$)

$$\equiv p_{i_1}(\overline{p_{i_2}p_{i_2+1}})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$+(\overline{p_{i_1}+p_{i_2}+p_{i_2+1}})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m$$

$$\equiv p_{i_1}(\overline{p_{i_2}p_{i_2+1}})\cdot\bar{p}_1\cdots\bar{p}_{i_1-1}\cdot\bar{p}_{i_1+1}\cdots\bar{p}_{i_2-1}\cdot\bar{p}_{i_2+2}\cdots\bar{p}_m+\overline{S}$$

Now, $S(\vec{t}) \neq I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{p_{i_2}p_{i_2+1}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m + \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in TP_{i_1}(S) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2, i_2+1}}^{m} TP_i(S) \big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S) \big) \Big)$, or
2. $\vec{t} \in FP(S)$.

Hence, the result follows. □

Theorem 4.16 *(TNF ⋈ DORF - Case 2)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the i_1-th term, p_{i_1}, is negated and the subexpression $(p_{i_2} + p_{i_2+1})$ is implemented as $p_{i_2}p_{i_2+1}$ where $1 \leq (i_1 = i_2) < m$, the resulting expression denoted as $I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ is equivalent to that given by Expression (16) in Table 1. Then, $S \not\equiv I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ if and only if there is a test case $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S) \big)$.

Proof : Since $i_1 = i_2$, we observe that $S \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$

$\equiv \big((p_{i_1} + p_{i_1+1}) \oplus (\bar{p}_{i_1}p_{i_1+1}) \big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+2} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_1+1}) \cdot (\overline{\bar{p}_{i_1}p_{i_1+1}}) + (\overline{p_{i_1} + p_{i_1+1}}) \cdot (\bar{p}_{i_1}p_{i_1+1}) \big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+2} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_1+1}) \cdot (\bar{\bar{p}}_{i_1} + \bar{p}_{i_1+1}) + (\overline{p_{i_1} + p_{i_1+1}}) \cdot (\bar{p}_{i_1}p_{i_1+1}) \big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+2} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_1+1}) \cdot (p_{i_1} +$

$barp_{i_1+1}) + (\overline{p_{i_1} + p_{i_1+1}}) \cdot (\bar{p}_{i_1}p_{i_1+1}) \big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+2} \cdots \bar{p}_m$

$\equiv (p_{i_1} + 0) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+2} \cdots \bar{p}_m$

(By making use of $(A+B)(A+\overline{B}) \equiv A$ and $(\overline{A+B})(\overline{A}B) \equiv 0$)

$\equiv p_{i_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+2} \cdots \bar{p}_m$

Now, $S(\vec{t}) \neq I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2} p_{i_2+1})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2} p_{i_2+1})}(\vec{t})$ = 1

if and only if $p_{i_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{i_1+2} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S) \big)$.

Hence, the result follows. □

Theorem 4.17 *(TNF ⋈ CORF - Case 1)*

Let $S=p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the i_1-th term, p_{i_1}, in S is negated and the i_2-th term, p_{i_2}, in S is implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $1 \leq i_1 < i_2 \leq m$, $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$ and $1 \leq j_2 < k_{i_2}$, the resulting expression denoted as $I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ is equivalent to that given by Expression (17) in Table 1. Then, $S \not\equiv I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$, or*
2. *$\vec{t} \in FP(S)$.*

Proof : First, we observe that $S \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$

$$\equiv \big((p_{i_1} + p_{i_2}) \oplus (\bar{p}_{i_1} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2}) \cdot (\overline{\bar{p}_{i_1} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) + (\overline{p_{i_1} + p_{i_2}}) \cdot (\bar{p}_{i_1} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\big)$$
$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2}) \cdot (\overline{\bar{p}_{i_1}} \cdot \overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) + (\overline{p_{i_1} + p_{i_2}}) \cdot (\bar{p}_{i_1} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\big)$$
$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + p_{i_2}) \cdot p_{i_1} \cdot \overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}} + (\overline{p_{i_1} + p_{i_2}}) \cdot (\bar{p}_{i_1} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\big)$$
$$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1}) \cdot (\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) + (\overline{p_{i_1} + p_{i_2}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

(By making use of $(\overline{A+BC})(\overline{A}+B+C) \equiv \overline{A+BC}$)

$$\equiv p_{i_1} \cdot (\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$+(\overline{p_{i_1} + p_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv p_{i_1}\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2,1,j_2} + \bar{p}_{i_2,j_2+1,k_{i_2}}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$$

(By rewriting AB as $AB(A+B)$ because they are equivalent)

$$\equiv p_{i_1}\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot (\overline{p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \overline{S}$$

$$\equiv p_{i_1}\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2} \cdots \bar{p}_m + \overline{S}$$

$$\equiv p_{i_1}\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + \overline{S}$$

$$\equiv p_{i_1}(\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + \overline{S}$$

Now, $S(\vec{t}) \neq I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$, or
2. $\vec{t} \in FP(S)$.

Hence, the result follows. □

Theorem 4.18 *(TNF ⋈ CORF - Case 2)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the i_1-th term, p_{i_1}, in S is negated and the i_2-th term, p_{i_2}, in S is implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $1 \leq (i_1 = i_2) \leq m$, $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$ and $1 \leq j_2 < k_{i_2}$, the resulting expression denoted as $I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$ is equivalent to that given by Expression (18) in Table 1. Then, $S \not\equiv I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_2}(S)$*, or*

2. $\vec{t} \in FP(S)$ *such that* $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ *on* $\vec{t}$*.*

Proof : Since $i_1 = i_2$, we observe that $S \oplus I_{TNF(p_{i_1} \to \bar{p}_{i_1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$

$\equiv \big((p_{i_2}) \oplus (\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_2}) \cdot (\overline{\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}}) + (\bar{p}_{i_2}) \cdot (\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_2}) \cdot (p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) + (\bar{p}_{i_2}) \cdot (\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big(p_{i_2} + \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,j_2+1,k_{i_2}}\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $(\overline{AB})(\overline{A} \cdot \overline{B}) \equiv \overline{A} \cdot \overline{B}$)

$\equiv p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2,1,j_2} + \bar{p}_{i_2,j_2+1,k_{i_2}}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By rewriting AB as $AB(A+B)$ because they are equivalent)

$\equiv p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot (\overline{p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_2} \cdots \bar{p}_m$

$\equiv p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + \bar{p}_{i_2,1,j_2} \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \overline{S}$

$\equiv p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) \cdot \overline{S}$

Now, $S(\vec{t}) \neq I_{TNF(p_{i_2} \to \bar{p}_{i_2}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$

if and only if $S(\vec{t}) \oplus I_{TNF(p_{i_2} \to \bar{p}_{i_2}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}(\vec{t}) = 1$

if and only if $p_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m + (\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) \cdot \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_2}(S)$, or

2. $\vec{t} \in FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ on $\vec{t}$.

Hence, the result follows. □

4.1.3 TOF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is a TOF. Since ENF ⋈ TOF and TNF ⋈ TOF have been discussed in previous sections, we do not repeat the discussions here.

Theorem 4.19 *(TOF ⋈ TOF)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two different terms, p_{i_1} and p_{i_2}, in S are omitted where $1 \leq i_1 < i_2 \leq m$, the resulting expression denoted as $I_{TOF(p_{i_1}\rightarrow\ )\bowtie TOF(p_{i_2}\rightarrow\ )}$ is equivalent to that given by Expression (19) in Table 1. Then, $S \not\equiv I_{TOF(p_{i_1}\rightarrow\ )\bowtie TOF(p_{i_2}\rightarrow\ )}$ if and only if there is a test case $\vec{t} \in \big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,i_2}}^{m} TP_i(S)\big)$.

Proof : First, we observe that $S \oplus I_{TOF(p_{i_1}\rightarrow\ )\bowtie TOF(p_{i_2}\rightarrow\ )}$

$\equiv S \oplus (p_1 + \ldots + p_{i_1-1} + p_{i_1+1} + p_{i_2-1} + p_{i_2+1} + \ldots + p_m)$

$\equiv (p_{i_1} + p_{i_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $(A+B) \oplus B \equiv A \cdot \overline{B}$)

Now,	$S(\vec{t}) \neq I_{TOF(p_{i_1}\rightarrow\ )\bowtie TOF(p_{i_2}\rightarrow\ )}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{TOF(p_{i_1}\rightarrow\ )\bowtie TOF(p_{i_2}\rightarrow\ )}(\vec{t}) = 1$
if and only if	$(p_{i_1} + p_{i_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t} \in \big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,i_2}}^{m} TP_i(S)\big)$.

Hence, the result follows. □

Theorem 4.20 *(TOF ⋈ DORF)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the i_1-th term, p_{i_1}, in S is omitted and the subexpression $(p_{i_2} + p_{i_2+1})$ in S is implemented as $p_{i_2}p_{i_2+1}$ where $1 \leq i_1 < i_2 < m$, the resulting expression denoted as

$I_{TOF(p_{i_1}\rightarrow\ )\bowtie DORF(p_{i_2}+p_{i_2+1}\rightarrow p_{i_2}p_{i_2+1})}$ *is equivalent to that given in Expression (20) in Table 1. Then,* $S \not\equiv I_{TOF(p_{i_1}\rightarrow\ )\bowtie DORF(p_{i_2}+p_{i_2+1}\rightarrow p_{i_2}p_{i_2+1})}$ *if and only if there is a test case* $\vec{t} \in \Big(\big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,i_2}}^{m} TP_i(S))\Big) \bigcup \Big(\big(TP_{i_1}(S) \cup TP_{i_2+1}(S)\big) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,i_2+1}}^{m} TP_i(S))\Big)$.

Proof : First, we observe that $S \oplus I_{TOF(p_{i_1}\rightarrow\ )\bowtie DORF(p_{i_2}+p_{i_2+1}\rightarrow p_{i_2}p_{i_2+1})}$

$\equiv \big((p_{i_1}+p_{i_2}+p_{i_2+1}) \oplus (p_{i_2}p_{i_2+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_2}+p_{i_2+1}) \cdot (\overline{p_{i_2}p_{i_2+1}}) + (\overline{p_{i_1}+p_{i_2}+p_{i_2+1}}) \cdot (p_{i_2}p_{i_2+1})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_2}+p_{i_2+1}) \cdot (\bar{p}_{i_2}+\bar{p}_{i_2+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$

(By making use of $(\overline{A+B+C})(BC) \equiv 0$)

$\equiv \big(\bar{p}_{i_2}(p_{i_1}+p_{i_2+1}) + \bar{p}_{i_2+1}(p_{i_1}+p_{i_2})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$

$\equiv (p_{i_1}+p_{i_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$+(p_{i_1}+p_{i_2+1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$

Now, $S(\vec{t}) \neq I_{TOF(p_{i_1}\rightarrow\ )\bowtie DORF(p_{i_2}+p_{i_2+1}\rightarrow p_{i_2}p_{i_2+1})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{TOF(p_{i_1}\rightarrow\ )\bowtie DORF(p_{i_2}+p_{i_2+1}\rightarrow p_{i_2}p_{i_2+1})}(\vec{t})$ = 1

if and only if $(p_{i_1}+p_{i_2}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$+(p_{i_1}+p_{i_2+1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in \Big(\big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,i_2}}^{m} TP_i(S))\Big) \bigcup \Big(\big(TP_{i_1}(S) \cup TP_{i_2+1}(S)\big) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,i_2+1}}^{m} TP_i(S))\Big)$

Hence, the result follows. □

Theorem 4.21 *(TOF ⋈ CORF)*

Let S=$p_1+\cdots+p_m$ *be a Boolean specification in irredundant disjunctive normal form. Suppose that the* i_1*-th term,* p_{i_1}*, in* S *is omitted and the* i_2*-th term,* p_{i_2}*, in* S *is implemented as* $p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}$ *where* $1 \leq i_1 < i_2 \leq m$*,* $1 \leq j_2 < k_{i_2}$ *and* $p_{i_2} = p_{i_2,1,j_2}p_{i_2,j_2+1,k_{i_2}}$*, the resulting expression denoted as*

$I_{TOF(p_{i_1}\rightarrow)\bowtie CORF(p_{i_2}\rightarrow p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ *is equivalent to that given by Expression (21) in Table 1. Then,* $S \not\equiv I_{TOF(p_{i_1}\rightarrow)\bowtie CORF(p_{i_2}\rightarrow p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ *if and only if there is a test case* $\vec{t}$ *that satisfies any of the following conditions:*

1. $\vec{t} \in UTP_{i_1}(S)$ *such that* $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ *on* $\vec{t}$*, or*

2. $\vec{t} \in FP(S)$ *such that* $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$ *on* $\vec{t}$*.*

Proof : First, we observe that $S \oplus I_{TOF(p_{i_1}\rightarrow)\bowtie CORF(p_{i_2}\rightarrow p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$

$\equiv \big((p_{i_1}+p_{i_2}) \oplus (p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_2}) \cdot (\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}}) + (\overline{p_{i_1}+p_{i_2}}) \cdot (p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_2}) \cdot (\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}}) + (\bar{p}_{i_1} \cdot \bar{p}_{i_2}) \cdot (p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv (p_{i_1}\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} + \bar{p}_{i_1}p_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} + \bar{p}_{i_1}\bar{p}_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}})$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $(A+B\cdot C)(\overline{B}\cdot\overline{C}) \equiv A\cdot\overline{B}\cdot\overline{C}$ and $(\overline{A+B\cdot C})(B+C) \equiv \overline{A}\cdot B\cdot\overline{C}+\overline{A}\cdot\overline{B}\cdot C$)

$\equiv p_{i_1}\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1}\bar{p}_{i_2+1} \cdots \bar{p}_m$

$+(p_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} + \bar{p}_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2,1,j_2} + \bar{p}_{i_2,j_2+1,k_{i_2}}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$+(p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1} \cdots \bar{p}_{i_2-1} \cdot (\overline{p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By rewriting AB as $AB(A+B)$ because they are equivalent;

and $(A\bar{B}+\bar{A}B)$ as $(A+B)\overline{AB}$ because they are equivalent)

$\equiv p_{i_1}\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot (\overline{p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$+(p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_2} \cdots \bar{p}_m$

$\equiv p_{i_1}\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2} \cdots \bar{p}_m$

$+(p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$

$\equiv p_{i_1}\bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + (p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$

$\equiv p_{i_1}(\overline{p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + (p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$

Now, $S(\vec{t}) \neq I_{TOF(p_{i_1}\rightarrow)\bowtie CORF(p_{i_2}\rightarrow p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{TOF(p_{i_1}\rightarrow)\bowtie CORF(p_{i_2}\rightarrow p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + (p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S)$ such that $p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}} = 0$ on $\vec{t}$, or
2. $\vec{t} \in FP(S)$ such that $p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}} = 1$ on $\vec{t}$.

Hence, the result follows. □

4.1.4 DORF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is a DORF. Since ENF ⋈ DORF, TNF ⋈ DORF and TOF ⋈ DORF have been discussed in previous sections, we do not repeat the discussions here.

Theorem 4.22 *(DORF ⋈ DORF - Case 1)*
Let $S=p_1+\cdots+p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two different subexpressions $(p_{i_1}+p_{i_1+1})$ and $(p_{i_2}+p_{i_2+1})$ in S are implemented as $p_{i_1}p_{i_1+1}$ and $p_{i_2}p_{i_2+1}$ respectively, where $1 \leq i_1 < i_1+1 < i_2 < m$, the resulting expression denoted as $I_{DORF(p_{i_1}+p_{i_1+1}\rightarrow p_{i_1}p_{i_1+1})\bowtie DORF(p_{i_2}+p_{i_2+1}\rightarrow p_{i_2}p_{i_2+1})}$ is equivalent to that given by Expression (22) in Table 1. Then, $S \not\equiv I_{DORF(p_{i_1}+p_{i_1+1}\rightarrow p_{i_1}p_{i_1+1})\bowtie DORF(p_{i_2}+p_{i_2+1}\rightarrow p_{i_2}p_{i_2+1})}$ if and only if there is a test case $\vec{t} \in \Big(\bigcup_{i=i_1,i_1+1,i_2,i_2+1} TP_i(S)\Big) \setminus \Big(\Big(\bigcup_{\substack{i=1 \\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)\Big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$.

Proof : First, we observe that $S \oplus I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1}p_{i_1+1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$

$\equiv \big((p_{i_1}+p_{i_1+1}+p_{i_2}+p_{i_2+1}) \oplus (p_{i_1}p_{i_1+1}+p_{i_2}p_{i_2+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_1+1}+p_{i_2}+p_{i_2+1}) \cdot (\overline{p_{i_1}p_{i_1+1}+p_{i_2}p_{i_2+1}}) + (\overline{p_{i_1}+p_{i_1+1}+p_{i_2}+p_{i_2+1}})$

$\cdot (p_{i_1}p_{i_1+1}+p_{i_2}p_{i_2+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$

$\equiv (p_{i_1}+p_{i_1+1}+p_{i_2}+p_{i_2+1})\overline{p_{i_1}p_{i_1+1}} \cdot \overline{p_{i_2}p_{i_2+1}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$

(By making use of $(\overline{A+B+C+D})(AB+CD) \equiv 0$)

Now, $S(\vec{t}) \neq I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1}p_{i_1+1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1}p_{i_1+1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}(\vec{t}) = 1$

if and only if $(p_{i_1}+p_{i_1+1}+p_{i_2}+p_{i_2+1})\overline{p_{i_1}p_{i_1+1}} \cdot \overline{p_{i_2}p_{i_2+1}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+2} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in \Big(\bigcup_{i=i_1,i_1+1,i_2,i_2+1} TP_i(S)\Big) \setminus \Big(\Big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)\Big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$.

Hence, the result follows. □

Theorem 4.23 *(DORF ⋈ DORF - Case 2)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the two subexpressions $(p_{i_1}+p_{i_1+1})$ and $(p_{i_2}+p_{i_2+1})$ in S are implemented as $p_{i_1}p_{i_1+1}$ and $p_{i_2}p_{i_2+1}$, respectively, where $1 \leq i_1 \leq m-2$ and $i_2 = i_1+1$, the resulting expression denoted as $I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1}p_{i_1+1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ is equivalent to that given by Expression (23) in Table 1. Then, $S \not\equiv I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1}p_{i_1+1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$ if and only if there is a test case $\vec{t} \in \Big(\bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S)\Big) \setminus \Big(\Big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1,i_1+2}}^{m} TP_i(S)\Big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)\Big)$.

Proof : Since $i_2 = i_1+1$, we observe that $S \oplus I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1}p_{i_1+1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \to p_{i_2}p_{i_2+1})}$

$\equiv \big((p_{i_1}+p_{i_1+1}+p_{i_1+2}) \oplus (p_{i_1}p_{i_1+1}p_{i_1+2})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m$

$\equiv \big((p_{i_1}+p_{i_1+1}+p_{i_1+2}) \cdot (\overline{p_{i_1}p_{i_1+1}p_{i_1+2}}) + (\overline{p_{i_1}+p_{i_1+1}+p_{i_1+2}}) \cdot (p_{i_1}p_{i_1+1}p_{i_1+2})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m$

$\equiv (p_{i_1} + p_{i_1+1} + p_{i_1+2})(\overline{p_{i_1} p_{i_1+1} p_{i_1+2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m$

(By making use of $\overline{(A+B+C)}(ABC) \equiv 0$)

Now, $S(\vec{t}) \neq I_{DORF(p_{i_1}+p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \rightarrow p_{i_2} p_{i_2+1})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{DORF(p_{i_1}+p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \bowtie DORF(p_{i_2}+p_{i_2+1} \rightarrow p_{i_2} p_{i_2+1})}(\vec{t}) = 1$

if and only if $(p_{i_1} + p_{i_1+1} + p_{i_1+2})(\overline{p_{i_1} p_{i_1+1} p_{i_1+2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+3} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in \Big(\bigcup_{i=i_1, i_1+1, i_1+2} TP_i(S) \Big) \setminus \Big(\Big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1, i_1+2}}^{m} TP_i(S) \Big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S) \big) \Big)$.

Hence, the result follows. □

Theorem 4.24 *(DORF ⋈ CORF - Case 1)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + p_{i_1+1})$ is implemented as $p_{i_1} p_{i_1+1}$ and the i_2-th term, p_{i_2}, is implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$, where $1 \leq i_1 < i_1+1 < i_2 \leq m$, $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$ and $1 \leq j_2 < k_{i_2}$, the resulting expression denoted as $I_{DORF(p_{i_1}+p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \bowtie CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$ is equivalent to Expression (24) in Table 1. Then, $S \not\equiv I_{DORF(p_{i_1}+p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \bowtie CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ on $\vec{t}$, or*

2. *$\vec{t} \in FP(S)$, such that such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$ on $\vec{t}$.*

Proof : First, we observe that $S \oplus I_{DORF(p_{i_1}+p_{i_1+1} \rightarrow p_{i_1} p_{i_1+1}) \bowtie CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$

$\equiv \big((p_{i_1} + p_{i_1+1} + p_{i_2}) \oplus (p_{i_1} p_{i_1+1} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_1+1} + p_{i_2})(\overline{p_{i_1} p_{i_1+1} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) + (\overline{p_{i_1} + p_{i_1+1} + p_{i_2}})$
$\cdot (p_{i_1} p_{i_1+1} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_1+1} + p_{i_2})(\bar{p}_{i_1} + \bar{p}_{i_1+1}) \cdot \bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} + (\bar{p}_{i_1} \cdot \bar{p}_{i_1+1} \cdot \bar{p}_{i_2})$

$$\cdot(p_{i_1}p_{i_1+1}+p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv ((p_{i_1}+p_{i_2})\bar{p}_{i_1+1}\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}}+(p_{i_1+1}+p_{i_2})\bar{p}_{i_1}\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}}+(\bar{p}_{i_1}\cdot\bar{p}_{i_1+1}\cdot\bar{p}_{i_2})$$

$$\cdot(p_{i_1}p_{i_1+1}+p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$$

$$\equiv (p_{i_1}\bar{p}_{i_1+1}\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}}+p_{i_1+1}\bar{p}_{i_1}\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1}$$

$$\cdot\bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1}\bar{p}_{i_1+1} \cdots \bar{p}_{i_2} \cdots \bar{p}_m$$

(By making use of $(C+A\cdot B)\overline{A}\cdot\overline{B} \equiv C\cdot\overline{A}\cdot\overline{B}$ and $\overline{A}\cdot\overline{B}(A\cdot B+C) \equiv \overline{A}\cdot\overline{B}\cdot C$)

$$\equiv (p_{i_1}\bar{p}_{i_1+1}+p_{i_1+1}\bar{p}_{i_1})\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot (\bar{p}_{i_2,1,j_2}+\bar{p}_{i_2,j_2+1,k_{i_2}})$$

$$\cdot\bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$$

(By rewriting AB as $AB(A+B)$ because they are equivalent)

$$\equiv (p_{i_1}\bar{p}_{i_1+1}+p_{i_1+1}\bar{p}_{i_1})\bar{p}_{i_2,1,j_2}\bar{p}_{i_2,j_2+1,k_{i_2}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2-1} \cdot (\overline{p_{i_2,1,j_2}p_{i_2,j_2+1,k_{i_2}}})$$

$$\cdot\bar{p}_{i_2+1} \cdots \bar{p}_m + (p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$$

$$\equiv (p_{i_1}\bar{p}_{i_1+1}+p_{i_1+1}\bar{p}_{i_1})(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_{i_2} \cdots \bar{p}_m$$

$$+(p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$$

$$\equiv p_{i_1}(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1+1}(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1}$$

$$\cdot\bar{p}_{i_1+2} \cdots \bar{p}_m + (p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$$

Now, $S(\vec{t}) \neq I_{DORF(p_{i_1}+p_{i_1+1}\to p_{i_1}p_{i_1+1}) \ \ltimes CORF(p_{i_2}\to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{DORF(p_{i_1}+p_{i_1+1}\to p_{i_1}p_{i_1+1}) \ \ltimes CORF(p_{i_2}\to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}(\vec{t}) = 1$

if and only if $p_{i_1}(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + p_{i_1+1}(\overline{p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + (p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ such that $p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}} = 0$ on $\vec{t}$, or
2. $\vec{t} \in FP(S)$, such that such that $p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}} = 1$ on $\vec{t}$.

Hence, the result follows. □

Theorem 4.25 *(DORF $\bowtie$ CORF - Case 2)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the subexpression $(p_{i_1} + p_{i_1+1})$ in S is implemented as $p_{i_1} p_{i_1+1}$ and the i_1-th term, p_{i_1}, in S is implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$, where $1 \le i_1 < m$, $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$ and $1 \le j_1 < k_{i_1}$, the resulting expression denoted as $I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1} p_{i_1+1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$ is equivalent to Expression (25) in Table 1. Then, $S \not\equiv I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1} p_{i_1+1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$ if and only if there is a test case $\vec{t}$ that satisfies any of the following conditions:

1. *$\vec{t} \in UTP_{i_1+1}(S)$, such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0$ on $\vec{t}$, or*
2. *$\vec{t} \in FP(S)$, such that $p_{i_1,1,j_1} = 1$ on $\vec{t}$.*

Proof : Since $i_2 = i_1$, we observe that $S \oplus I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1} p_{i_1+1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$

$\equiv \big((p_{i_1} + p_{i_1+1}) \oplus (p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} p_{i_1+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_1+1})(\overline{p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} p_{i_1+1}}) + (\overline{p_{i_1} + p_{i_1+1}})(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} p_{i_1+1})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

$\equiv (\bar{p}_{i_1,1,j_1} \bar{p}_{i_1,j_1+1,k_{i_1}} p_{i_1+1} + p_{i_1,1,j_1} \bar{p}_{i_1,j_1+1,k_{i_1}} \bar{p}_{i_1+1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

(By making use of $(A \cdot B + C)(\overline{A + B \cdot C}) \equiv \overline{A} \cdot \overline{B} \cdot C$ and $(\overline{A \cdot B + C})(A + B \cdot C) \equiv A \cdot \overline{B} \cdot \overline{C}$)

$\equiv \bar{p}_{i_1,1,j_1} \bar{p}_{i_1,j_1+1,k_{i_1}} p_{i_1+1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

$+ p_{i_1,1,j_1} \bar{p}_{i_1,j_1+1,k_{i_1}} \bar{p}_{i_1+1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

$\equiv \bar{p}_{i_1,1,j_1} \bar{p}_{i_1,j_1+1,k_{i_1}} p_{i_1+1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\bar{p}_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}}) \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

$+ p_{i_1,1,j_1} \bar{p}_{i_1,j_1+1,k_{i_1}} \bar{p}_{i_1+1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\overline{p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}}) \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

(By rewriting AB as $AB(A+B)$ because they are equivalent;

and $A\overline{B}$ as $A \cdot \overline{(AB)}$ because they are equivalent)

$\equiv \bar{p}_{i_1,1,j_1} \bar{p}_{i_1,j_1+1,k_{i_1}} p_{i_1+1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + p_{i_1,1,j_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1} \cdots \bar{p}_m$

$\equiv p_{i_1+1} \bar{p}_{i_1,1,j_1} \bar{p}_{i_1,j_1+1,k_{i_1}} \cdot \bar{p}_1 \cdots \bar{p}_{i_1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + p_{i_1,1,j_1} \cdot \overline{S}$

$\equiv p_{i_1+1} (\overline{p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + p_{i_1,1,j_1} \cdot \overline{S}$

Now,	$S(\vec{t}) \neq I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1} p_{i_1+1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{DORF(p_{i_1}+p_{i_1+1} \to p_{i_1} p_{i_1+1}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}(\vec{t}) = 1$
if and only if	$p_{i_1+1}(\overline{p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + p_{i_1,1,j_1} \cdot \overline{S}$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t}$ satisfies any of the following conditions:

1. $\vec{t} \in UTP_{i_1+1}(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0$ on $\vec{t}$, or

2. $\vec{t} \in FP(S)$, such that $p_{i_1,1,j_1} = 1$ on $\vec{t}$.

Hence, the result follows. $\square$

4.1.5 CORF with Other Faults

In this section, we study the detection conditions of double faults in which one of the single fault is a CORF. Since ENF $\bowtie$ CORF, TNF $\bowtie$ CORF, TOF $\bowtie$ CORF and DORF$\bowtie$ CORF have been discussed in previous sections, we do not repeat the discussions here.

Theorem 4.26 *(CORF $\bowtie$ CORF - Case 1)*
Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two different terms p_{i_1} and p_{i_2} in S are implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ and $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ respectively, where $1 \leq i_1 < i_2 \leq m$, $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$, $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$, $1 \leq j_1 < k_{i_1}$ and $1 \leq j_2 < k_{i_2}$, the resulting expression denoted as $I_{CORF(p_{i_1} \to p_{i_1,1,j_1}+p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ is equivalent to that given by Expression (26) in Table 1. Then, $S \not\equiv I_{CORF(p_{i_1} \to p_{i_1,1,j_1}+p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2}+p_{i_2,j_2+1,k_{i_2}})}$ if and only if there is a test case $\vec{t} \in FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$ on $\vec{t}$.

Proof : First, we observe that $S \oplus I_{CORF(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$

$\equiv \big((p_{i_1} + p_{i_2}) \oplus (p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})(\overline{p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}}) + (\overline{p_{i_1} + p_{i_2}})$

$\cdot (p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1} + p_{i_2})\bar{p}_{i_1,1,j_1} \cdot \bar{p}_{i_1,j_1+1,k_{i_1}} \cdot \bar{p}_{i_2,1,j_2} \cdot \bar{p}_{i_2,j_2+1,k_{i_2}} + (\bar{p}_{i_1} \cdot \bar{p}_{i_2})$

$\cdot (p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

$\equiv (p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})\bar{p}_{i_1}\bar{p}_{i_2} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_{i_2-1} \cdot \bar{p}_{i_2+1} \cdots \bar{p}_m$

(By making use of $(AB+CD)(\overline{A} \cdot \overline{B} \cdot \overline{C} \cdot \overline{D}) \equiv 0$)

$\equiv (p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$

Now,	$S(\vec{t}) \neq I_{CORF(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}(\vec{t})$
if and only if	$S(\vec{t}) \oplus I_{CORF(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}(\vec{t}) = 1$
if and only if	$(p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}) \cdot \overline{S}$ evaluates to 1 on $\vec{t}$
if and only if	$\vec{t} \in FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$ on $\vec{t}$

Hence, the result follows. □

Theorem 4.27 *(CORF ⋈ CORF - Case 2)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that two different CORFs are committed at a particular term in S. That is, the i_1-th term, p_{i_1}, in S is implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,j_2} \cdot p_{i_1,j_2+1,k_{i_1}}$ and $1 \leq j_1 < j_2 < k_{i_1}$, the resulting expression denoted as $I_{CORF(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$ is equivalent to Expression (27) in Table 1. Then, $S \not\equiv I_{CORF(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$ if and only if there is a test case $\vec{t} \in FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1$ on $\vec{t}$.

Proof : First, we observe that $S \oplus I_{CORF(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \rightarrow p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}$

$\equiv \big((p_{i_1}) \oplus (p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \left(p_{i_1}(\overline{p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}}}) + \overline{p_{i_1}}(p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}})\right)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv (p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}})\bar{p}_{i_1} \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$

(By making use of $(ABC)(\overline{A+B+C}) \equiv 0$)

$\equiv (p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}}) \cdot \overline{S}$

Now, $S(\vec{t}) \neq I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie CORF(p_{i_2} \to p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}})}(\vec{t}) = 1$

if and only if $(p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}}) \cdot \overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}}$ =1 on $\vec{t}$.

Hence, the result follows. □

4.2 Dectection Conditions of 4 Remaining Faulty Implementations

In this section, we study the detection conditions of 4 double-fault expressions in double faults with ordering which do not have their equivalent counterparts in double faults without ordering.

Theorem 4.28 *(TOF ⋈ DORF)*

Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the i_1-th term, p_{i_1}, is omitted and then the subexpression $(p_{i_1-1} + p_{i_1+1})$ is implemented as $p_{i_1-1}p_{i_1+1}$ where $1 < i_1 < m$, the resulting expression denoted as $I_{TOF(p_{i_1}\to) \bowtie DORF(p_{i_1-1}+p_{i_1+1} \to p_{i_1-1}p_{i_1+1})}$ is equivalent to that given by Expression (53) in Table 1. Then, we have $S \not\equiv I_{TOF(p_{i_1}\to) \bowtie DORF(p_{i_1-1}+p_{i_1+1} \to p_{i_1-1}p_{i_1+1})}$ if and only if there is a test case $\vec{t} \in$ $\Big((TP_{i_1-1}(S) \cup TP_{i_1}(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1-1, i_1}}^{m} TP_i(S))\Big) \bigcup \Big((TP_{i_1}(S) \cup TP_{i_1+1}(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S))\Big)$.

Proof : First, we observe that

$S \oplus I_{TOF(p_{i_1}\to) \bowtie DORF(p_{i_1-1}+p_{i_1+1} \to p_{i_1-1}p_{i_1+1})}$

$= \big((p_{i_1-1}+p_{i_1}+p_{i_1+1}) \oplus (p_{i_1-1}p_{i_1+1})\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-2} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

$= \big((p_{i_1-1}+p_{i_1}+p_{i_1+1})(\overline{p_{i_1-1}p_{i_1+1}}) + (\overline{p_{i_1-1}+p_{i_1}+p_{i_1+1}})(p_{i_1-1}p_{i_1+1})\big)$

$\cdot \bar{p}_1 \cdots \bar{p}_{i_1-2} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

$= \big(\bar{p}_{i_1-1}(p_{i_1}+p_{i_1+1}) + \bar{p}_{i_1+1}(p_{i_1}+p_{i_1-1}) + 0\big) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-2} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

(By making use of $(A+B+C)(\overline{AC}) \equiv \overline{A}(B+C) + \overline{B}(A+C)$ and $\overline{(A+B+C)}(AC) \equiv 0$)

$= (p_{i_1}+p_{i_1+1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m + (p_{i_1-1}+p_{i_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-2} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$

$= (p_{i_1-1}+p_{i_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-2} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + (p_{i_1}+p_{i_1+1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$

Now, $S(\vec{t}) \neq I_{TOF(p_{i_1}\rightarrow) \bowtie DORF(p_{i_1-1}+p_{i_1+1}\rightarrow p_{i_1-1}p_{i_1+1})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{TOF(p_{i_1}\rightarrow\ ) \bowtie DORF(p_{i_1-1}+p_{i_1+1}\rightarrow p_{i_1-1}p_{i_1+1})}(\vec{t}) = 1$

if and only if $(p_{i_1-1} + p_{i_1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-2} \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m + (p_{i_1} + p_{i_1+1}) \cdot \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot \bar{p}_{i_1+2} \cdots \bar{p}_m$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in \Big(\big(TP_{i_1-1}(S) \cup TP_{i_1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1-1,i_1}}^{m} TP_i(S)\big)\Big) \bigcup \Big(\big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1,i_1+1}}^{m} TP_i(S)\big)\Big)$.

Hence, the result follows. □

Theorem 4.29 *(CORF with ENF)*

Let $S=p_1+\cdots+p_m$ *be a Boolean specification in irredundant disjunctive normal form. Suppose the* i_1*-th term,* p_{i_1}*, in* S *is implemented as* $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ *where* $1 \leq i_1 < h_1 < m$*,* $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$ *and* $1 \leq j_1 < k_{i_1}$*, and the subexpression* $p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}$ *is then negated, the resulting expression denoted as* $I_{CORF(p_{i_1}\rightarrow p_{i_1,1,j_1}+p_{i_1,j_1+1,k_{i_1}}) \bowtie ENF(p_{i_1,j_1+1,k_{i_1}}+\cdots+p_{h_1}\rightarrow \overline{p_{i_1,j_1+1,k_{i_1}}+\cdots+p_{h_1}})}$ *is equivalent to that given by Expression (70) in Table 1. Then, we have* $S \neq I_{CORF(p_{i_1}\rightarrow p_{i_1,1,j_1}+p_{i_1,j_1+1,k_{i_1}}) \bowtie ENF(p_{i_1,j_1+1,k_{i_1}}+\cdots+p_{h_1}\rightarrow \overline{p_{i_1,j_1+1,k_{i_1}}+\cdots+p_{h_1}})}$ *if and only if there is a test case* $\vec{t}$ *that satisfies any of the following conditions*

1. $\vec{t} \in \big(\bigcup_{i=i_1+1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1+1,\ldots,h_1}}^{m} TP_i(S)\big)$ *such that* $p_{i_1,1,j_1} = 0$ *on* $\vec{t}$*, or*

2. $\vec{t} \in FP(S)$ such that $p_{i_1,j_1+1,k_{i_1}} = 0$ on $\vec{t}$.

Proof : First, we observe that

$$S \oplus I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie ENF(p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1} \to \overline{p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}})}$$

$$\equiv \big((p_{i_1} + \cdots + p_{h_1}) \oplus (p_{i_1,1,j_1} + \overline{p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}})\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + \cdots + p_{h_1})(\overline{p_{i_1,1,j_1} + \overline{p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}}}) + (\overline{p_{i_1} + \cdots + p_{h_1}})(p_{i_1,1,j_1}$$

$$+ \overline{p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}})\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + \cdots + p_{h_1})(\bar{p}_{i_1,1,j_1} \cdot \overline{\overline{p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}}}) + \bar{p}_{i_1} \cdot \overline{p_{i_1+1} + \cdots + p_{h_1}}(p_{i_1,1,j_1}$$

$$+ \bar{p}_{i_1,j_1+1,k_{i_1}} \cdot \overline{p_{i_1+1} + \cdots + p_{h_1}})\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1} + \cdots + p_{h_1}) \bar{p}_{i_1,1,j_1}(p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}) + \overline{p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}} \cdot \overline{p_{i_1+1} + \cdots + p_{h_1}}$$

$$\cdot (p_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}} \cdot \overline{p_{i_1+1} + \cdots + p_{h_1}})\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$\equiv \big((p_{i_1+1} + \cdots + p_{h_1}) \bar{p}_{i_1,1,j_1}(p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_{h_1}) + (\bar{p}_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}})$$

$$\cdot \overline{p_{i_1+1} + \cdots + p_{h_1}} \cdot (p_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}} \overline{p_{i_1+1} + \cdots + p_{h_1}})\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m$$

(By making use of $(AB + C)\overline{A} \equiv \overline{A}C$)

$$\equiv \big(\bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) + \bar{p}_{i_1,j_1+1,k_{i_1}} \cdot \overline{p_{i_1+1} + \cdots + p_{h_1}}\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m$$

(By making use of $(\overline{A} + \overline{B})C(A + \overline{B}C) \equiv \overline{B}C$)

$$\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$+ (\bar{p}_{i_1,j_1+1,k_{i_1}} \cdot \overline{p_{i_1+1} + \cdots + p_{h_1}})\ \bar{p}_1 \cdots \bar{p}_{i_1-1} \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\bar{p}_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}}) \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$+ \bar{p}_{i_1,j_1+1,k_{i_1}}\ \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\bar{p}_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}}) \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$$

(By rewriting A as $A(A+B)$ because they are equivalent;

and B as $B(A+B)$ because they are equivalent)

$$\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\overline{p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}}) \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m$$

$$+ \bar{p}_{i_1,j_1+1,k_{i_1}}\ \bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\overline{p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}}) \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$$

$$\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \bar{p}_1 \cdots \bar{p}_{i_1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \bar{p}_{i_1,j_1+1,k_{i_1}} \bar{p}_1 \cdots \bar{p}_{i_1} \cdots \bar{p}_m$$

$$\equiv \bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1}) \bar{p}_1 \cdots \bar{p}_{i_1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \bar{p}_{i_1,j_1+1,k_{i_1}} \overline{S}$$

Now, $S(\vec{t}) \neq I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie ENF(p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1} \to \overline{p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie ENF(p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1} \to \overline{p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}})}(\vec{t})$ = 1

if and only if $\bar{p}_{i_1,1,j_1}(p_{i_1+1} + \cdots + p_{h_1})\bar{p}_1 \cdots \bar{p}_{i_1} \cdot \bar{p}_{h_1+1} \cdots \bar{p}_m + \bar{p}_{i_1,j_1+1,k_{i_1}}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t}$ satisfies any of the following conditions

1. $\vec{t} \in \big(\bigcup_{i=i_1+1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1+1,\ldots,h_1}}^{m} TP_i(S)\big)$ such that $p_{i_1,1,j_1} = 0$ on $\vec{t}$, or
2. $\vec{t} \in FP(S)$ such that $p_{i_1,j_1+1,k_{i_1}} = 0$ on $\vec{t}$.

Hence, the result follows. □

Theorem 4.30 *(CORF with TNF)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the i_1-th term, p_{i_1}, in S is implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ where $1 \leq i_1 \leq m$, $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$ and $1 \leq j_1 < k_{i_1}$, and the term $p_{i_1,j_1+1,k_{i_1}}$ is then negated, the expression denoted as $I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TNF(p_{i_1,j_1+1,k_{i_1}} \to \bar{p}_{i_1,j_1+1,k_{i_1}})}$ is equivalent to that given by Expression (73) in Table 1. Then, $S \not\equiv I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TNF(p_{i_1,j_1+1,k_{i_1}} \to \bar{p}_{i_1,j_1+1,k_{i_1}})}$ if and only if there is a test case $\vec{t} \in FP(S)$ such that $p_{i_1,j_1+1,k_{i_1}} = 0$ on $\vec{t}$.

Proof : First, we observe that

$$S \oplus I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TNF(p_{i_1,j_1+1,k_{i_1}} \to \bar{p}_{i_1,j_1+1,k_{i_1}})}$$
$$\equiv \big((p_{i_1}) \oplus (p_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1})(\overline{p_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}}}) + (\bar{p}_{i_1})(p_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}})\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$$
$$\equiv \big((p_{i_1})(\bar{p}_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}) + \bar{p}_{i_1}(p_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}})\big)\ \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$$
$$\equiv (\bar{p}_{i_1,j_1+1,k_{i_1}})\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$$

(By making use of $AB(\overline{A}B) \equiv 0$ and $(\overline{AB})(A + \overline{B}) \equiv \overline{B}$)

$\equiv (\bar{p}_{i_1,j_1+1,k_{i_1}})\bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\bar{p}_{i_1,1,j_1} + \bar{p}_{i_1,j_1+1,k_{i_1}}) \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$

(By rewriting A as $A(A+B)$ because they are equivalent)

$\equiv (\bar{p}_{i_1,j_1+1,k_{i_1}})\bar{p}_1 \cdots \bar{p}_{i_1-1} \cdot (\overline{p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}}) \cdot \bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv (\bar{p}_{i_1,j_1+1,k_{i_1}})\ \bar{p}_1 \cdots \bar{p}_{i_1} \cdots \bar{p}_m$

$\equiv (\bar{p}_{i_1,j_1+1,k_{i_1}})\overline{S}$

Now, $S(\vec{t}) \neq I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TNF(p_{i_1,j_1+1,k_{i_1}} \to \bar{p}_{i_1,j_1+1,k_{i_1}})}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TNF(p_{i_1,j_1+1,k_{i_1}} \to \bar{p}_{i_1,j_1+1,k_{i_1}})}(\vec{t}) = 1$

if and only if $(\bar{p}_{i_1,j_1+1,k_{i_1}})\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in FP(S)$ such that $p_{i_1,j_1+1,k_{i_1}} = 0$ on $\vec{t}$.

Hence, the result follows. □

Theorem 4.31 *(CORF with TOF)*

Let $S = p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Suppose that the i_1-th term, p_{i_1}, in S is implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ where $1 \leq i_1 \leq m$, $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$ and $1 \leq j_1 < k_{i_1}$, and then the term $p_{i_1,j_1+1,k_{i_1}}$ in S is omitted from the expression, the resulting expression denoted as $I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TOF(p_{i_1,j_1+1,k_{i_1}} \to)}$ is equivalent to that given by Expression (76) in Table 1. Then, $S \not\equiv I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TOF(p_{i_1,j_1+1,k_{i_1}} \to)}$ if and only if there is a test case $\vec{t} \in FP(S)$ such that $p_{i_1,1,j_1} = 1$ on $\vec{t}$.

Proof : First, we observe that

$S \oplus I_{CORF(p_{i_1} \to p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TOF(p_{i_1,j_1+1,k_{i_1}} \to)}$

$\equiv \big((p_{i_1}) \oplus (p_{i_1,1,j_1})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \big((p_{i_1})(\bar{p}_{i_1,1,j_1}) + (\bar{p}_{i_1})(p_{i_1,1,j_1})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \big(0 + (\bar{p}_{i_1})(p_{i_1,1,j_1})\big)\bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv \bar{p}_{i_1} p_{i_1,1,j_1} \bar{p}_1 \cdots \bar{p}_{i_1-1}\bar{p}_{i_1+1} \cdots \bar{p}_m$

$\equiv p_{i_1,1,j_1}\bar{p}_1 \cdots \bar{p}_{i_1} \cdots \bar{p}_m$

$\equiv p_{i_1,1,j_1}\overline{S}$

Now, $S(\vec{t}) \neq I_{CORF(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TOF(p_{i_1,j_1+1,k_{i_1}} \rightarrow)}(\vec{t})$

if and only if $S(\vec{t}) \oplus I_{CORF(p_{i_1} \rightarrow p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}) \bowtie TOF(p_{i_1,j_1+1,k_{i_1}} \rightarrow)}(\vec{t}) = 1$

if and only if $p_{i_1,1,j_1}\overline{S}$ evaluates to 1 on $\vec{t}$

if and only if $\vec{t} \in FP(S)$ such that $p_{i_1,1,j_1}$ evaluate to 1.

Hence, the result follows. □

For ease of reading and understanding, we list all double fault classes, the corresponding double-fault expression numbers and their corresponding fault detection conditions in Table 2. Let us consider the third row of Table 2, which presents the detection conditions of two double-fault expressions of ENF ⋈ TOF. For double-fault expression (5) (please refer to Table 1 for the actual double-fault expression), the detection condition shows that any true point of S in $(\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S))$ or any false point of S can distinguish S and the expression. While for double-fault expression (6) in Table 1, the detection condition shows that any true point of S in $(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S))$ or any false point of S can distinguish S and the expression.

5 Fault Coupling

Most fault-based testing strategies derive their test cases based on the assumption that at most one of the hypothesized faults is committed by programmers [11]. A fundamental question in fault-based testing is whether test cases that detect programs with single fault in isolation are also able to detect programs with multiple faults in combination.

Fault coupling has been studied for years [5, 12], but so far it has no universally agreed definition. In this report, faults are said to be *coupled* together if they can be detected in isolation but not

Table 2: Double fault, double-fault expression and detection condition ($S = p_1 + \ldots + p_m$)

(a) Double-fault expressions (1)–(12) due to double faults without ordering

Fault Class	(Expression No.) : Detection Condition
ENF ⋈ ENF	(1) : (C1) any point in $\Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \cap \big(\bigcup_{i=i_2}^{h_2} TP_i(S)\big)\Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2,\ldots,h_2}}^{m} TP_i(S)\big)$, or (C2) any point in $FP(S)$
	(2) : (C1) any point in $\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$
ENF ⋈ TNF	(3) : (C1) any point in $\Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \cap \big(TP_{i_2}(S)\big)\Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)\big)$, or (C2) any point in $FP(S)$
	(4) : (C1) any point in $\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$
ENF ⋈ TOF	(5) : (C1) any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)\big)$, or (C2) any point in $FP(S)$
	(6) : (C1) any point in $\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$, or (C2) any point in $FP(S)$
ENF ⋈ DORF	(7) : (C1) any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$, or (C2) any point in $FP(S)$
	(8) : (C1) any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1+1}}^{m} TP_i(S)\big)$
	(9) : (C1) any point in $\big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1}}^{m} TP_i(S)\big)$, or (C2) any point in $FP(S)$
	(10) : (C1) any point in $\Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2,i_2+1}}^{h_1} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$, or (C2) any point in $FP(S)$
ENF ⋈ CORF	(11) : (C1) any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$, or (C2) any point in $FP(S)$
	(12) : (C1) any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$, or (C2) any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$

Table 2 (cont'd): Double fault, double-fault expression and detection condition

(a) Double-fault expressions (13)–(27) due to double faults without ordering

Fault Class	(Expression No.) : Detection Condition
TNF ⋈ TNF	(13) : (C1) any point in $\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S)\big)$, or (C2) any point in $FP(S)$
TNF ⋈ TOF	(14) : (C1) any point in $\big(TP_{i_1}(S)\big) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S)\big)$, or (C2) any point in $FP(S)$
TNF ⋈ DORF	(15) : (C1) any point in $TP_{i_1}(S) \setminus \Big(\big(\bigcup_{i=1, i \neq i_1,i_2,i_2+1}^{m} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$, or (C2) any point in $FP(S)$
	(16) : (C1) any point in $TP_{i_1}(S) \setminus \big(\bigcup_{i=1, i \neq i_1,i_1+1}^{m} TP_i(S)\big)$
TNF ⋈ CORF	(17) : (C1) any point in $UTP_{i_1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$, or (C2) any point in $FP(S)$
	(18) : (C1) any point in $UTP_{i_2}(S)$, or (C2) any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$
TOF ⋈ TOF	(19) : (C1) any point in $\big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S)\big)$
TOF ⋈ DORF	(20) : (C1) any point in $\Big(\big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2}^{m} TP_i(S)\big)\Big) \bigcup \Big(\big(TP_{i_1}(S) \cup TP_{i_2+1}(S)\big) \setminus \big(\bigcup_{i=1, i \neq i_1,i_2+1}^{m} TP_i(S)\big)\Big)$
TOF ⋈ CORF	(21) : (C1) any point in $UTP_{i_1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$, or (C2) any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$
DORF ⋈ DORF	(22) : (C1) any point in $\big(\bigcup_{i=i_1,i_1+1,i_2,i_2+1} TP_i(S)\big) \setminus \Big(\big(\bigcup_{i=1, i \neq i_1,i_1+1,i_2,i_2+1}^{m} TP_i(S)\big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$
	(23) : (C1) any point in $\big(\bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S)\big) \setminus \Big(\big(\bigcup_{i=1, i \neq i_1,i_1+1,i_1+2}^{m} TP_i(S)\big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)\Big)$
DORF ⋈ CORF	(24) : (C1) any point in $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$, or (C2) any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$
	(25) : (C1) any point in $UTP_{i_1+1}(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0$, or (C2) any point in $FP(S)$ such that $p_{i_1,1,j_1} = 1$
CORF ⋈ CORF	(26) : (C1) any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$
	(27) : (C1) any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1$

Table 2 (cont'd): Double fault, double-fault expression and detection condition

(b) Four double-fault expressions (53), (70), (73) and (76) due to double fault with ordering

Fault Class	(Expression No.) : Detection Condition
TOF ⋈ DORF	(53) : (C1) any point in $\Big(\big(TP_{i_1}(S) \cup TP_{i_1-1}(S)\big) \setminus \big(\bigcup_{i=1, i\neq i_1-1,i_1}^{m} TP_i(S)\big)\Big) \bigcup \Big(\big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{i=1, i\neq i_1,i_1+1}^{m} TP_i(S)\big)\Big)$
CORF ⋈ ENF	(70) : (C1) any point in $\big(\bigcup_{i=i_1+1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{i=1, i\neq i_1+1,\ldots,h_1}^{m} TP_i(S)\big)$ such that $p_{i_1,1,j_1} = 0$, or (C2) any point in $FP(S)$ such that $p_{i_1,j_1+1,k_{i_1}} = 0$
CORF ⋈ TNF	(73) : (C1) any point in $FP(S)$ such that $p_{i_1,j_1+1,k_{i_1}} = 0$
CORF ⋈ TOF	(76) : (C1) any point in $FP(S)$ such that $p_{i_1,1,j_1} = 1$

when combined together. Most previous studies on fault coupling focused on the combination of two single faults.

An empirical study on fault coupling via mutation analysis was done in [12]. A *mutant* is a program which differs from the original program by small syntactic changes. A *1-order* (respectively, *2-order*) mutant is a mutant that differs from the original program by 1 syntactic change (respectively, 2 syntactic changes). Three programs whose size ranges from 16 to 28 lines of code (LOC) were studied. Test sets that can kill all 1-order mutants were generated and used to kill 2-order mutants. As mentioned in [12], the experiment studies the *mutation* coupling effect, to be precise. It was found that test sets so generated can kill approximately 99.9% of 2-order mutants. It was then concluded that the effect of two faults being coupled together rarely occurs.

How Tai Wah [5] investigated fault coupling from a theoretical perspective. He also studied the behaviour of double faults. A program is modelled as a composition of several mathematical functions. For example, suppose that a program P is considered as a composition of three mathematical functions f, g and h in the order of f being computed first, followed by g and finally h, then P is equivalent to the composite function $h \circ g \circ f$. A single fault in a program is modelled as an incorrect use of one of the functions during the composition. For example, if a fault occurs in the program P, it is possible that any one of the three functions f, g or h is implemented wrongly as f', g' or h', respectively, to result in a faulty program which may be equivalent to $h \circ g \circ f'$, $h \circ g' \circ f$ or $h' \circ g \circ f$.

Moreover, a double fault in a program is modelled as the incorrect use of any two functions during the composition. Test sets that detect individual faults of a double fault are called *proper test sets*. Among the proper test sets, those that cannot detect the double fault are called *coupled test sets*. How Tai Wah [5] then calculates the *coupling ratio*, defined as the ratio of the number of coupled test sets to that of proper test sets. He shows analytically that the coupling ratio is approximately $1/|D|$ for test sets of size 1 and $1/|D|^2$ for test sets of size 2, where $|D|$ is the size of the input domain D. It should be noted that when a test set of size 1 detects a double fault, the only test case in the set must be able to detect each of the two individual faults in isolation. As $|D|$ is usually very large, the coupling ratio is very small, and he concludes that fault coupling rarely occurs.

In this study, we are more interested to know which double fault class can be detected by test cases that can detect the individual single fault classes. Hence, instead of performing empirical study via mutation analysis or calculating the coupling ratio via mathematical analysis, we analyse the relationship between single and double faults based on their detection conditions. More precisely, if DC_A, DC_B and $DC_{A \bowtie B}$ are the detection conditions of the single fault classes A and B and the double fault class $A \bowtie B$, respectively, we would like to identify the relationship among DC_A, DC_B and $DC_{A \bowtie B}$. Our aim is to find out which double fault class $A \bowtie B$ can be detected by test cases that can detect either of the two single fault classes A and B.

For test cases that detect fault class A, there are three mutually exclusive possibilities:

(R1) All points that satisfy DC_A will also satisfy $DC_{A \bowtie B}$; that is, any point that can detect A will also detect $A \bowtie B$.

(R2) Some but not all points that satisfy DC_A also satisfy $DC_{A \bowtie B}$; that is, some but not all points that can detect A can also detect $A \bowtie B$.

(R3) None of the points that satisfy DC_A also satisfies $DC_{A \bowtie B}$; that is, none of the points that can detect A will also detect $A \bowtie B$.

Similarly, for test cases that can detect B, there are three other mutually exclusive possibilities:

(R4) All points that satisfy DC_B will also satisfy $DC_{A \bowtie B}$; that is, any point that can detect B will also detect $A \bowtie B$.

(R5) Some but not all points that satisfy DC_B also satisfy $DC_{A \bowtie B}$; that is, some but not all points that can detect B can also detect $A \bowtie B$.

(R6) None of the points that satisfy DC_B also satisfies $DC_{A \bowtie B}$; that is, none of the points that can detect B will also detect $A \bowtie B$.

The analysis of the relationships between single faults and the 27 faulty implementations of double fault without ordering based on detection conditions are presented in Section 5.1 whereas those of the remaining 4 faulty implementations are introduced in Section 5.2.

5.1 Fault Coupling on 27 Faulty Implementations

5.1.1 ENF and ENF

In this section, we analyse the relationship between the double fault involving two ENFs. Since there are two ENFs, we use ENF1 and ENF2 to identify the two different ENFs in double fault ENF1$\bowtie$ENF2 for ease of understanding

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{h_1}$ $(i_1 < h_1)$ in S is wrongly negated, the corresponding detection condition, denoted by DC_{ENF1}, is "any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use TC_{ENF1} to denote the set of all points that satisfy DC_{ENF1}, that is $TC_{ENF1} = \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)\Big) \cup FP(S)$.

Similarly, if the subexpression $p_{i_2} + \cdots + p_{h_2}$ $(i_2 < h_2)$ in S is wrongly negated, the corresponding detection condition, denoted by DC_{ENF2}, is "any point in $\big(\bigcup_{i=i_2}^{h_2} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_2,\ldots,h_2}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use TC_{ENF2} to denote the set of all points that satisfy DC_{ENF2}, that is $TC_{ENF2} = \Big(\big(\bigcup_{i=i_2}^{h_2} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_2,\ldots,h_2}}^{m} TP_i(S)\big)\Big) \cup FP(S)$.

For ENF1 $\bowtie$ ENF2, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (1) in Table 1 where $i_1 < h_1 < i_2 < h_2$. The detection condition, denoted by $DC_{ENF1\bowtie ENF2-1}$, is "any point in $\Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \cap \big(\bigcup_{i=i_2}^{h_2} TP_i(S)\big)\Big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2,\ldots,h_2}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use $TC_{ENF1\bowtie ENF2-1}$ to denote the set of all points that satisfy $DC_{ENF1\bowtie ENF2-1}$, that is $TC_{ENF1\bowtie ENF2-1} = \bigg(\Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \cap \big(\bigcup_{i=i_2}^{h_2} TP_i(S)\big)\Big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2,\ldots,h_2}}^{m} TP_i(S)\big)\bigg) \cup FP(S)$.

(A) Relationship between DC_{ENF1} and $DC_{ENF1\bowtie ENF2-1}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{ENF1} can satisfy $DC_{ENF1\bowtie ENF2-1}$. For example, in TC_{ENF1}, false points of S (that is, points in $FP(S)$) satisfy $DC_{ENF1\bowtie ENF2-1}$.

(b) Some points in TC_{ENF1} cannot satisfy $DC_{ENF1\bowtie ENF2-1}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{ENF1}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_1}(S) = \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1\\ i\neq i_1}}^{m} TP_i(S)\big)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF1\bowtie ENF2-1}$ because $\vec{t} \notin FP(S)$, $\vec{t} \notin \bigcup_{i=i_2}^{h_2} TP_i(S)$ and $i_1 < i_2 < h_2$.

Hence, only some, but not all, points satisfying DC_{ENF1} can satisfy $DC_{ENF1\bowtie ENF2-1}$.

(B) Relationship between DC_{ENF2} and $DC_{ENF1\ltimes ENF2-1}$: The relationship R5 holds because of the following reasons:

(a) Some points in TC_{ENF2} can satisfy $DC_{ENF1\ltimes ENF2-1}$. For example, in TC_{ENF2}, false points of S (that is, points in $FP(S)$) satisfy $DC_{ENF1\ltimes ENF2-1}$.

(b) Some points in TC_{ENF2} cannot satisfy $DC_{ENF1\ltimes ENF2-1}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{ENF2}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_2}(S) = \big(TP_{i_2}(S) \setminus \bigcup_{\substack{i=1\\i\neq i_2}}^{m} TP_i(S)\big)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF1\ltimes ENF2-1}$ because $\vec{t} \notin FP(S)$, $\vec{t} \notin \bigcup_{i=i_1}^{h_1} TP_i(S)$ and $i_1 < h_1 < i_2$.

Hence, only some, but not all, points satisfying DC_{ENF2} can satisfy $DC_{ENF1\ltimes ENF2-1}$.

Case 2 The double-fault expression is equivalent to Expression (2) in Table 1 where $\{i_2,\ldots,h_2\} \subsetneqq \{i_1,\ldots,h_1\}$. The detection condition, denoted by $DC_{ENF1\ltimes ENF2-2}$, is "any point in $\big(\bigcup_{\substack{i=i_1\\i\neq i_2,\ldots,h_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$". We use $TC_{ENF1\ltimes ENF2-2}$ to denote the set of all points that satisfy $DC_{ENF1\ltimes ENF2-2}$, that is $TC_{ENF1\ltimes ENF2-2} = \big(\bigcup_{\substack{i=i_1\\i\neq i_2,\ldots,h_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$.

(A) Relationship between DC_{ENF1} and $DC_{ENF1\ltimes ENF2-2}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{ENF1} can satisfy $DC_{ENF1\ltimes ENF2-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. Since $\{i_2,\ldots,h_2\} \subsetneqq \{i_1,\ldots,h_1\}$, there exists $i_3 \in \{i_1,\ldots,h_1\}$ such that $i_3 \notin \{i_2,\ldots,h_2\}$. Now, in TC_{ENF1}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_3}(S) = \big(TP_{i_3}(S) \setminus \bigcup_{\substack{i=1\\i\neq i_3}}^{m} TP_i(S)\big)$ and $\vec{t}$ satisfies $DC_{ENF1\ltimes ENF2-2}$ because $\vec{t} \in \big(TP_{i_3}(S) \setminus \bigcup_{\substack{i=1\\i\neq i_3}}^{m} TP_i(S)\big) \subseteq \Big(\big(\bigcup_{\substack{i=i_1\\i\neq i_2,\ldots,h_2}}^{h_1} TP_i(S)\big) \setminus$

$\Big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S))\Big) = TC_{ENF1 \ltimes ENF2-2}$ (by using $A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B$).

(b) Some points in TC_{ENF1} cannot satisfy $DC_{ENF1 \ltimes ENF2-2}$. For example, in TC_{ENF1}, false points of S (that is, points in $FP(S)$) do not satisfy $DC_{ENF1 \ltimes ENF2-2}$.

Hence, only some, but not all, points satisfying DC_{ENF1} can satisfy $DC_{ENF1 \ltimes ENF2-2}$.

(B) Relationship between DC_{ENF2} and $DC_{ENF1 \ltimes ENF2-2}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{ENF2} \cap TC_{ENF1 \ltimes ENF2-2} \\
= & \Bigg(\Big((\bigcup_{i=i_2}^{h_2} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_2,\ldots,h_2}}^{m} TP_i(S)) \Big) \cup FP(S) \Bigg) \cap \Big((\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big) \\
= & \Bigg(\Big((\bigcup_{i=i_2}^{h_2} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_2,\ldots,h_2}}^{m} TP_i(S)) \Big) \cap \Big((\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big) \Bigg) \\
& \cup \Bigg(FP(S) \cap \Big((\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big) \Bigg) \\
= & \Bigg((\bigcup_{i=i_2}^{h_2} TP_i(S)) \setminus \Big((\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,i_2,\ldots,h_2,\ldots,h_1}}^{m} TP_i(S)) \cup (\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)) \Big) \Bigg) \\
& \cap \Big((\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big) \cup \emptyset \\
\subseteq & \Big((\bigcup_{i=i_2}^{h_2} TP_i(S)) \setminus (\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)) \Big) \cap \Big((\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big)
\end{aligned}
$$

(By using $A \setminus (B \cup C) \subseteq (A \setminus C)$)

$= \emptyset$ (By using $(A \setminus B) \cap (B \setminus C) = \emptyset$).

Hence, all points satisfying DC_{ENF2} do not satisfy $DC_{ENF1 \ltimes ENF2-2}$.

5.1.2 ENF and TNF

In this section, we analyse the relationship between the double fault ENF⋉TNF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{h_1}$ ($i_1 < h_1$) in S is wrongly negated, the corresponding detection condition, denoted by DC_{ENF}, is "any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\dots,h_1}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use TC_{ENF} to denote the set of all points that satisfy DC_{ENF}, that is $TC_{ENF} = \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\dots,h_1}}^{m} TP_i(S)\big)\Big) \cup FP(S)$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly negated, the corresponding detection condition, denoted by DC_{TNF}, is "any point in $UTP_{i_2}(S)$ or any point in $FP(S)$". We use TC_{TNF} to denote the set of all points that satisfy DC_{TNF}, that is $TC_{TNF} = UTP_{i_2}(S) \cup FP(S)$.

For ENF⋈TNF, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (3) in Table 1 where $i_1 < h_1 < i_2$. The detection condition, denoted by $DC_{ENF \bowtie TNF-1}$, is "any point in $\Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \cap TP_{i_2}(S)\Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\dots,h_1,i_2}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use $TC_{ENF \bowtie TNF-1}$ to denote the set of all points that satisfy $DC_{ENF \bowtie TNF-1}$, that is $TC_{ENF \bowtie TNF-1} = \bigg(\Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \cap TP_{i_2}(S)\Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\dots,h_1,i_2}}^{m} TP_i(S)\big)\bigg) \cup FP(S)$.

(A) Relationship between DC_{ENF} and $DC_{ENF \bowtie TNF-1}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{ENF} can satisfy $DC_{ENF \bowtie TNF-1}$. For example, in TC_{ENF}, false points of S (that is, points in $FP(S)$) satisfy $DC_{ENF \bowtie TNF-1}$.

(b) Some points in TC_{ENF} cannot satisfy $DC_{ENF \bowtie TNF-1}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{ENF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_1}(S) = \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \bowtie TNF-1}$ because $\vec{t} \notin FP(S)$, $\vec{t} \notin TP_{i_2}(S)$

and $i_1 < i_2$.

Hence, only some, but not all, points satisfying DC_{ENF} can satisfy $DC_{ENF \rtimes TNF-1}$.

(B) Relationship between DC_{TNF} and $DC_{ENF \rtimes TNF-1}$: The relationship R5 holds because of the following reasons:

(a) Some points in TC_{TNF} can satisfy $DC_{ENF \rtimes TNF-1}$. For example, in TC_{TNF}, false points of S (that is, points in $FP(S)$) satisfy $DC_{ENF \rtimes TNF-1}$.

(b) Some points in TC_{TNF} cannot satisfy $DC_{ENF \rtimes TNF-1}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{TNF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_2}(S) = \big(TP_{i_2}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_2}}^{m} TP_i(S)\big)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \rtimes TNF-1}$ because $\vec{t} \notin FP(S), \vec{t} \notin \bigcup_{i=i_1}^{h_1} TP_i(S)$ and $i_1 < h_1 < i_2$.

Hence, only some, but not all, points satisfying DC_{TNF} can satisfy $DC_{ENF \rtimes TNF-1}$.

Case 2 The double-fault expression is equivalent to Expression (4) in Table 1 where $i_1 \leq i_2 \leq h_1$ and $i_1 < h_1$. The detection condition, denoted by $DC_{ENF \rtimes TNF-2}$, is "any point in $\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$". We use $TC_{ENF \rtimes TNF-2}$ to denote the set of all points that satisfy $DC_{ENF \rtimes TNF-2}$, that is $TC_{ENF \rtimes TNF-2} = \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$.

(A) Relationship between DC_{ENF} and $DC_{ENF \rtimes TNF-2}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{ENF} can $DC_{ENF \rtimes TNF-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. Since $i_1 \leq i_2 \leq h_1$ and $i_1 \neq h_1$, there exists i_3 such that $i_1 \leq i_3 \leq h_1$ and $i_2 \neq i_3$. Now, in TC_{ENF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_3}(S) = \big(TP_{i_3}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_3}}^{m} TP_i(S)\big)$ and $\vec{t}$ satisfies

$DC_{ENF \bowtie TNF-2}$ because $\vec{t} \in \big(TP_{i_3}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_3}}^{m} TP_i(S)\big) \subseteq \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big) =$ $TC_{ENF \bowtie TNF-2}$ (by using $A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B$).

(b) Some points in TC_{ENF} cannot $DC_{ENF \bowtie TNF-2}$. For example, in TC_{ENF}, false points of S (that is, points in $FP(S)$) do not satisfy $DC_{ENF \bowtie TNF-2}$.

Hence, only some, but not all, points satisfying DC_{ENF} can satisfy $DC_{ENF \bowtie TNF-2}$.

(B) Relationship between DC_{TNF} and $DC_{ENF \bowtie TNF-2}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{TNF} \cap TC_{ENF \bowtie TNF-2} \\
= \; & \Big(UTP_{i_2}(S) \cup FP(S)\Big) \cap \Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)\Big) \\
= \; & \Big(UTP_{i_2}(S) \cap \Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)\Big)\Big) \\
& \cup \Big(FP(S) \cap \Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)\Big)\Big) \\
\subseteq \; & \Big(UTP_{i_2}(S) \cap \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S)\big)\Big) \cup \emptyset \qquad \text{(By using } A \cap (B \setminus C) \subseteq A \cap B\text{)} \\
= \; & \emptyset \qquad \text{(By definition of } UTP_{i_2}(S)\text{).}
\end{aligned}
$$

Hence, all points satisfying DC_{TNF} do not satisfy $DC_{ENF \bowtie TNF-2}$.

5.1.3 ENF and TOF

In this section, we analyse the relationship between the double fault ENF⋈TOF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{h_1}$ ($i_1 < h_1$) in S is wrongly negated, the corresponding detection condition, denoted by DC_{ENF}, is "any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use TC_{ENF} to denote the set of all points that satisfy DC_{ENF}, that is $TC_{ENF} = \Big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big) \cup FP(S)$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly omitted, the corresponding detection condition, denoted by DC_{TOF}, is "any point in $UTP_{i_2}(S)$". We use TC_{TOF} to denote the set of all points that satisfy DC_{TOF}, that is $TC_{TOF} = UTP_{i_2}(S)$.

For ENF⋈TOF, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (5) in Table 1 where $i_1 < h_1 < i_2$. The detection condition, denoted by $DC_{ENF \bowtie TOF-1}$, is "any point in $(\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S))$ or any point in $FP(S)$". We use $TC_{ENF \bowtie TOF-1}$ to denote the set of all points that satisfy $DC_{ENF \bowtie TOF-1}$, that is $TC_{ENF \bowtie TOF-1} = \Big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)) \Big) \cup FP(S)$.

(A) Relationship between DC_{ENF} and $DC_{ENF \bowtie TOF-1}$: The relationship R1 holds because:

$$
\begin{aligned}
& TC_{ENF} \\
= \; & \Big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big) \cup FP(S) \\
= \; & \Big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)) \cup TP_{i_2}(S) \Big) \Big) \cup FP(S) \\
\subseteq \; & \Big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)) \Big) \cup FP(S) \qquad \text{(By using } A \setminus (B \cup C) \subseteq A \setminus B\text{)} \\
= \; & TC_{ENF \bowtie TOF-1}.
\end{aligned}
$$

Hence, any point satisfying DC_{ENF} satisfies $DC_{ENF \bowtie TOF-1}$.

(B) Relationship between DC_{TOF} and $DC_{ENF \bowtie TOF-1}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{TOF} \cap TC_{ENF \ltimes TOF-1} \\
= \; & UTP_{i_2}(S) \cap \Big(\big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)) \big) \cup FP(S) \Big) \\
= \; & \Big(UTP_{i_2}(S) \cap \big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)) \big) \Big) \cup \Big(UTP_{i_2}(S) \cap FP(S) \Big) \\
\subseteq \; & \Big(UTP_{i_2}(S) \cap (\bigcup_{i=i_1}^{h_1} TP_i(S)) \Big) \cup \emptyset \qquad \text{(By using } A \setminus B \subseteq A\text{)} \\
= \; & \emptyset \qquad \text{(By definition of } UTP_{i_2}(S) \text{ and } i_1 < h_1 < i_2\text{).}
\end{aligned}
$$

Hence, all points satisfying DC_{TOF} do not satisfy $DC_{ENF \ltimes TOF-1}$.

Case 2 The double-fault expression is equivalent to Expression (6) in Table 1 where $i_1 \leq i_2 \leq h_1$ and $i_1 < h_1$. The detection condition, denoted by $DC_{ENF \ltimes TOF-2}$, is "any point in $\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S) \big)$ or any point in $FP(S)$". We use $TC_{ENF \ltimes TOF-2}$ to denote the set of points that satisfy $DC_{ENF \ltimes TOF-2}$, that is $TC_{ENF \ltimes TOF-2} = \Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S) \big) \Big) \cup FP(S)$.

(A) Relationship between DC_{ENF} and $DC_{ENF \ltimes TOF-2}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{ENF} can satisfy $DC_{ENF \ltimes TOF-2}$. For example, in TC_{ENF}, false points of S (that is, points in $FP(S)$) satisfy $DC_{ENF \ltimes TOF-2}$.

(b) Some points in TC_{ENF} cannot satisfy $DC_{ENF \ltimes TOF-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{ENF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_2}(S) = \big(TP_{i_2}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_2}}^{m} TP_i(S) \big)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \ltimes TOF-2}$ because $\vec{t} \notin FP(S)$

and $\vec{t} \notin \big(\bigcup_{\substack{i=i_1\\ i\neq i_2}}^{h_1} TP_i(S)\big)$.

Hence, only some, but not all, points satisfying DC_{ENF} can satisfy $DC_{ENF\ltimes TOF-2}$.

(B) Relationship between DC_{TOF} and $DC_{ENF\ltimes TOF-2}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{TOF} \cap TC_{ENF\ltimes TOF-2} \\
= \;& UTP_{i_2}(S) \cap \Big(\big((\bigcup_{\substack{i=i_1\\ i\neq i_2}}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)) \big) \cup FP(S) \Big) \\
= \;& \Big(UTP_{i_2}(S) \cap \big((\bigcup_{\substack{i=i_1\\ i\neq i_2}}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)) \big) \Big) \cup \Big(UTP_{i_2}(S) \cap FP(S) \Big) \\
\subseteq \;& \Big(UTP_{i_2}(S) \cap (\bigcup_{\substack{i=i_1\\ i\neq i_2}}^{h_1} TP_i(S)) \Big) \cup \emptyset \qquad \text{(By using } A \setminus B \subseteq A) \\
= \;& \emptyset \qquad \text{(By definition of } UTP_{i_2}(S)).
\end{aligned}
$$

Hence, all points satisfying DC_{TOF} do not satisfy $DC_{ENF\ltimes TOF-2}$.

5.1.4 ENF and DORF

In this section, we analyse the relationship between the double fault ENF⋉DORF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{h_1}$ ($i_1 < h_1$) in S is wrongly negated, the corresponding detection condition, denoted by DC_{ENF}, is "any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use TC_{ENF} to denote the set of all points that satisfy DC_{ENF}, that is $TC_{ENF} = \Big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big) \cup FP(S)$.

Similarly, if the subexpression $p_{i_2} + p_{i_2+1}$ in S is wrongly implemented as $p_{i_2} \cdot p_{i_2+1}$, the corresponding detection condition, denoted by DC_{DORF}, is "any point in $UTP_{i_2}(S)$ or any point in

$UTP_{i_2+1}(S)$". We use TC_{DORF} to denote the set of all points that satisfy DC_{DORF}, that is $TC_{DORF} = UTP_{i_2}(S) \cup UTP_{i_2+1}(S)$.

For ENF⋈DORF, there are four subcases.

Case 1 The double-fault expression is equivalent to Expression (7) in Table 1 where $i_1 < h_1 < i_2$.

The detection condition, denoted by $DC_{ENF \bowtie DORF-1}$, is "any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$ or any point in $FP(S)$". We use $TC_{ENF \bowtie DORF-1}$ to denote the set of all points that satisfy $DC_{ENF \bowtie DORF-1}$, that is $TC_{ENF \bowtie DORF-1} = \Bigg(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)\Bigg) \cup FP(S)$.

(A) Relationship between DC_{ENF} and $DC_{ENF \bowtie DORF-1}$: The relationship R1 holds because

$$
\begin{aligned}
& TC_{ENF} \\
= \;& \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)\Big) \cup FP(S) \\
= \;& \Bigg(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup TP_{i_2}(S) \cup TP_{i_2+1}(S)\Big)\Bigg) \cup FP(S) \\
\subseteq \;& \Bigg(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)\Bigg) \cup FP(S) \\
& \text{(By using } A \setminus \big(B \cup (C \cup D)\big) \subseteq A \setminus \big(B \cup (C \cap D)\big)\text{)} \\
= \;& TC_{ENF \bowtie DORF-1}.
\end{aligned}
$$

Hence, any point satisfying DC_{ENF} satisfies $DC_{ENF \bowtie DORF-1}$.

(B) Relationship between DC_{DORF} and $DC_{ENF \bowtie DORF-1}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{DORF} \cap TC_{ENF \ltimes DORF-1} \\
= \; & \big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \Bigg(\bigg(\Big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus \Big((\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2,i_2+1}}^{m} TP_i(S)) \\
& \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)\bigg) \cup FP(S)\Bigg) \\
= \; & \Bigg(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \bigg((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus \Big((\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2,i_2+1}}^{m} TP_i(S)) \\
& \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)\bigg)\Bigg) \cup \Big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap FP(S)\Big) \\
\subseteq \; & \Big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap (\bigcup_{i=i_1}^{h_1} TP_i(S))\Big) \cup \emptyset \qquad \text{(By using } A \setminus B \subseteq A\text{)} \\
= \; & \big(UTP_{i_2}(S) \cap \bigcup_{i=i_1}^{h_1} TP_i(S)\big) \cup \big(UTP_{i_2+1}(S) \cap \bigcup_{i=i_1}^{h_1} TP_i(S)\big) \\
= \; & \emptyset \qquad \text{(By definitions of } UTP_{i_2}(S) \text{ and } UTP_{i_2+1}(S)\text{, and } i_1 < h_1 < i_2\text{).}
\end{aligned}
$$

Hence, all points satisfying DC_{DORF} do not satisfy $DC_{ENF \ltimes DORF-1}$.

Case 2 The double-fault expression is equivalent to Expression (8) in Table 1 where $i_1 < h_1 = i_2$. The detection condition, denoted by $DC_{ENF \ltimes DORF-2}$, is "any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1+1}}^{m} TP_i(S)\big)$". We use $TC_{ENF \ltimes DORF-2}$ to denote the set of points that satisfy $DC_{ENF \ltimes DORF-2}$, that is $TC_{ENF \ltimes DORF-2} = \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1+1}}^{m} TP_i(S)\big)$.

(A) Relationship between DC_{ENF} and $DC_{ENF \ltimes DORF-2}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{ENF} can satisfy $DC_{ENF \ltimes DORF-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{ENF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_1}(S) = \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big)$. Note that, $\vec{t}$ satisfies $DC_{ENF \ltimes DORF-2}$ because $\vec{t} \in \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big) \subseteq$

$\Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1+1}}^{m} TP_i(S) \big) \Big) = TC_{ENF \bowtie DORF-2}$ (by using $A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B$).

(b) Some points in TC_{ENF} cannot satisfy $DC_{ENF \bowtie DORF-2}$. For example, in TC_{ENF}, false points of S (that is, points in $FP(S)$) do not satisfy $DC_{ENF \bowtie DORF-2}$.

Hence, only some, but not all, points satisfying DC_{ENF} can satisfy $DC_{ENF \bowtie DORF-2}$.

(B) Relationship between DC_{DORF} and $DC_{ENF \bowtie DORF-2}$: The relationship R5 holds because of the following reasons:

(a) Some points in TC_{DORF} can satisfy $DC_{ENF \bowtie DORF-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{DORF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_2}(S) = \big(TP_{i_2}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_2}}^{m} TP_i(S) \big)$. Note that, $\vec{t}$ satisfies $DC_{ENF \bowtie DORF-2}$ because $\vec{t} \in \big(TP_{i_2}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_2}}^{m} TP_i(S) \big) \subseteq \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1+1}}^{m} TP_i(S) \big) \Big) = TC_{ENF \bowtie DORF-2}$ (by using $A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B$ and $i_2 = h_1$).

(b) Some points in TC_{DORF} cannot satisfy $DC_{ENF \bowtie DORF-2}$. For example, in TC_{DORF}, we can find a point $\vec{t} \in UTP_{i_2+1}(S) = \big(TP_{i_2+1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_2+1}}^{m} TP_i(S) \big)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \bowtie DORF-2}$ because $\vec{t} \notin \big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big)$ and $i_2 = h_1$.

Hence, only some, but not all, points satisfying DC_{DORF} can satisfy $DC_{ENF \bowtie DORF-2}$.

Case 3 The double-fault expression is equivalent to Expression (9) in Table 1 where $i_1 = i_2$ and $h_1 = i_2 + 1$. The detection condition, denoted by $DC_{ENF \bowtie DORF-3}$, is "any point in $\big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1}}^{m} TP_i(S) \big)$ or any point in $FP(S)$". We use $TC_{ENF \bowtie DORF-3}$ to denote the set of all points that satisfy $DC_{ENF \ltimes DORF-3}$, that is $TC_{ENF \ltimes DORF-3} = \Big(\big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S) \big) \Big) \cup FP(S)$.

(A) Relationship between DC_{ENF} and $DC_{ENF \ltimes DORF-3}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{ENF} can satisfy $DC_{ENF \ltimes DORF-3}$. For example, in TC_{ENF}, false points of S (that is, points in $FP(S)$) satisfy $DC_{ENF \ltimes DORF-3}$.

(b) Some points in TC_{ENF} cannot satisfy $DC_{ENF \ltimes DORF-3}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{ENF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_1}(S) = \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S) \big)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \ltimes DORF-3}$ because $\vec{t} \notin TP_{i_1+1}(S)$.

Hence, only some, but not all, points satisfying DC_{ENF} can satisfy $DC_{ENF \ltimes DORF-3}$.

(B) Relationship between DC_{DORF} and $DC_{ENF \ltimes DORF-3}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{DORF} \cap TC_{ENF \ltimes DORF-3} \\
= \; & \big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \Bigg(\Big(\big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)\Big) \cup FP(S)\Bigg) \\
= \; & \Bigg(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \Big(\big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)\Big)\Bigg) \\
& \cup \Big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap FP(S)\Big) \\
= \; & \Bigg(UTP_{i_2}(S) \cap \Big(\big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)\Big)\Bigg) \\
& \cup \Bigg(UTP_{i_2+1}(S) \cap \Big(\big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)\Big)\Bigg) \cup \emptyset \\
\subseteq \; & \Big(UTP_{i_2}(S) \cap \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big)\Big) \cup \Big(UTP_{i_2+1}(S) \cap \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big)\Big) \\
& \quad \text{(By using } A \setminus B \subseteq A\text{)} \\
= \; & \big(UTP_{i_2}(S) \cap TP_{i_1+1}(S)\big) \cup \big(UTP_{i_2+1}(S) \cap TP_{i_1}(S)\big) \\
& \quad \text{(When } i_1 = i_2\text{, } UTP_{i_2}(S) \subseteq TP_{i_1}(S) \text{ and } UTP_{i_2+1}(S) \subseteq TP_{i_1+1}(S)\text{)} \\
= \; & \emptyset \quad \text{(By definitions of } UTP_{i_2}(S) \text{ and } UTP_{i_2+1}(S)\text{, and } i_1 = i_2\text{).}
\end{aligned}
$$

Hence, all points satisfying DC_{DORF} do not satisfy $DC_{ENF \ltimes DORF-3}$.

Case 4 The double-fault expression is equivalent to Expression (10) in Table 1 where $i_1 \leq i_2 < h_1$ and $h_1 \neq i_1 + 1$. The detection condition, denoted by $DC_{ENF \ltimes DORF-4}$, is "any point in $\Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2, i_2+1}}^{h_1} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, \ldots, h_1}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use $TC_{ENF \ltimes DORF-4}$ to denote the set of all points that satisfy $DC_{ENF \ltimes DORF-4}$, that is $TC_{ENF \ltimes DORF-4} = \Bigg(\Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2, i_2+1}}^{h_1} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, \ldots, h_1}}^{m} TP_i(S)\big)\Bigg) \cup FP(S)$.

(A) Relationship between DC_{ENF} and $DC_{ENF \ltimes DORF-4}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{ENF} can satisfy $DC_{ENF \ltimes DORF-4}$. For example, in TC_{ENF}, false points of

S (that is, points in $FP(S)$) satisfy $DC_{ENF \ltimes DORF-4}$.

(b) Some points in TC_{ENF} cannot satisfy $DC_{ENF \ltimes DORF-4}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{ENF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_2}(S) = \big(TP_{i_2}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_2}}^{m} TP_i(S)\big)$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \ltimes DORF-4}$ because $\vec{t} \notin TP_{i_2+1}(S)$.

Hence, only some, but not all, points satisfying DC_{ENF} can satisfy $DC_{ENF \ltimes DORF-4}$.

(B) Relationship between DC_{DORF} and $DC_{ENF \ltimes DORF-4}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{DORF} \cap TC_{ENF \ltimes DORF-4} \\
= \; & \big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \Bigg(\bigg(\Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2, i_2+1}}^{h_1} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big) \\
& \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, \ldots, h_1}}^{m} TP_i(S)\big)\bigg) \cup FP(S)\Bigg) \\
\subseteq \; & \big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \bigg(\Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2, i_2+1}}^{h_1} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big) \\
& \cup FP(S)\bigg) \quad \text{(By using } A \setminus B \subseteq A\text{)} \\
= \; & \bigg(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2, i_2+1}}^{h_1} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)\bigg) \\
& \cup \Big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap FP(S)\Big) \\
= \; & \bigg(\Big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2, i_2+1}}^{h_1} TP_i(S)\big)\Big) \cup \Big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \\
& \cap \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)\bigg) \cup \emptyset \quad \big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \subseteq UTP(S) \text{ and } UTP(S) \cap \\
& FP(S) = \emptyset\big) \\
= \; & \Big(UTP_{i_2}(S) \cap \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2, i_2+1}}^{h_1} TP_i(S)\big)\Big) \cup \Big(UTP_{i_2+1}(S) \cap \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2, i_2+1}}^{h_1} TP_i(S)\big)\Big) \cup \emptyset \quad \big(\big(UTP_{i_2}(S) \cup \\
& UTP_{i_2+1}(S)\big) \subseteq UTP(S), \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big) \subseteq OTP(S) \text{ and } UTP(S) \cap OTP(S) = \emptyset\,\big) \\
= \; & \emptyset \quad \text{(By definitions of } UTP_{i_2}(S) \text{ and } UTP_{i_2+1}(S)\text{)}.
\end{aligned}
$$

Hence, all points satisfying DC_{DORF} do not satisfy $DC_{ENF \ltimes DORF-4}$.

5.1.5 ENF and CORF

In this section, we analyse the relationship between the double fault ENF⋈CORF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + \cdots + p_{h_1}$ $(i_1 < h_1)$ in S is wrongly negated, the corresponding detection condition, denoted by DC_{ENF}, is "any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use TC_{ENF} to denote the set of all points that satisfy DC_{ENF}, that is $TC_{ENF} = \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)\Big) \cup FP(S)$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$, the corresponding detection condition, denoted by DC_{CORF}, is "any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ = 1". We use TC_{CORF} to denote the set of all points that satisfy DC_{CORF}, that is $TC_{CORF} = \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$.

For ENF⋈CORF, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (11) in Table 1 where $i_1 < h_1 < i_2$. The detection condition, denoted by $DC_{ENF \bowtie CORF-1}$, is "any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ or any point in $FP(S)$". We use $TC_{ENF \bowtie CORF-1}$ to denote the set of all points that satisfy $DC_{ENF \bowtie CORF-1}$, that is $TC_{ENF \bowtie CORF-1} = \{\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S)\big) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup FP(S)$.

(A) Relationship between DC_{ENF} and $DC_{ENF \bowtie CORF-1}$:

First, $TC_{ENF \bowtie CORF-1} = \{\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup$

$FP(S) \subseteq \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S) \big) \Big) \cup FP(S)) = TC_{ENF}$. Hence the relationship between DC_{ENF} and $DC_{ENF \bowtie CORF-1}$ depends on whether $TC_{ENF \bowtie CORF-1} = TC_{ENF}$.

If $TC_{ENF \bowtie CORF-1}$ and TC_{ENF} are not equal, R2 holds. That means, some points in TC_{ENF} do not satisfy $DC_{ENF \bowtie CORF-1}$. Here is an example. Let $S = ab + cd + ef$. If the subexpression $ab + cd$ is negated and the third term ef is wrongly split as $e + f$, $I_{ENF} = \overline{ab+cd} + ef$ and $I_{ENF \bowtie CORF-1} = \overline{ab+cd} + e + f$. Now, the point 110001 $\in UTP_1(S) = TP_1(S) \setminus \big(\bigcup_{i=2}^{6} TP_i(S) \big) \subseteq TC_{ENF}$ can distinguish S from I_{ENF} because S and I_{ENF} evaluate to 1 and 0 on this point, respectively. However, it cannot be used to distinguish S from $I_{ENF \bowtie CORF-1}$ because both S and $I_{ENF \bowtie CORF-1}$ evaluate to 1 on 110001.

On the other hand, if $TC_{ENF \bowtie CORF-1} = TC_{ENF}$, R1 holds. That means, $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ evaluates to 0 on all points in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S) \big)$. Under such circumstances, all test cases that can detect $ENF \bowtie CORF-1$ can also be used to detect ENF. Here is an example. Let $S = abc + de + \bar{a}\bar{b}d + c\bar{d} + \bar{a}e + \bar{b}e$. Note that $\big(TP_1(S) \cup TP_2(S) \big) \setminus \big(\bigcup_{i=3}^{6} TP_i(S) \big) = \{11110, 11011, 11111\}$. If the subexpression $abc + de$ is negated and the third term is wrongly split as $\bar{a} + \bar{b}d$, $I_{ENF} = \overline{abc+de} + \bar{a}\bar{b}d + c\bar{d} + \bar{a}e + \bar{b}e$ and $I_{ENF \bowtie CORF-1} = \overline{abc+de} + \bar{a} + \bar{b}d + c\bar{d} + \bar{a}e + \bar{b}e$. For ease of reference, let $X = \Big(\big(TP_1(S) \cup TP_2(S) \big) \setminus \big(\bigcup_{i=3}^{6} TP_i(S) \big) \Big)$. Note that $X = \{11110, 11011, 11111\}$, and $TC_{ENF} = X \cup FP(S) = \{11110, 11011, 11111\} \cup FP(S)$, and $TC_{ENF \bowtie CORF-1} = \{\vec{t} \in X : \bar{a} + \bar{b}d = 0\} \cup FP(S) = \{11110, 11011, 11111\} \cup FP(S) = TC_{ENF}$ because $\bar{a} + \bar{b}d = 0$ on all points in X.

(B) Relationship between DC_{CORF} and $DC_{ENF \bowtie CORF-1}$: The relationship R4 holds because:

$$
\begin{aligned}
& TC_{CORF} \\
= \; & \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \\
\subseteq \; & FP(S) \\
\subseteq \; & \{\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S) \big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1,i_2}}^{m} TP_i(S) \big) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup FP(S) \\
= \; & TC_{ENF \bowtie CORF-1}.
\end{aligned}
$$

Hence, any point satisfying DC_{CORF} satisfies $DC_{ENF \bowtie CORF-1}$.

Case 2 The double-fault expression is equivalent to Expression (12) in Table 1 where $i_1 \le i_2 \le h_1$ and $i_1 < h_1$. The detection condition, denoted by $DC_{ENF \bowtie CORF-2}$, is "any point in $\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$ or any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$". We use $TC_{ENF \bowtie CORF-2}$ to denote the set of all points that satisfy $DC_{ENF \bowtie CORF-2}$, that is $TC_{ENF \bowtie CORF-2} = \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)\Big) \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\}$.

(A) Relationship between DC_{ENF} and $DC_{ENF \bowtie CORF-2}$: The relationship R2 holds because of the following reasons:

  (a) Some points in TC_{ENF} can satisfy $DC_{ENF \bowtie CORF-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{ENF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_1}(S) = \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1\\ i\neq i_1}}^{m} TP_i(S)\big)$. Note that, $\vec{t}$ satisfies $DC_{ENF \bowtie CORF-2}$ because $\vec{t} \in \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1\\ i\neq i_1}}^{m} TP_i(S)\big) \subseteq \Big(\big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)\Big) \subseteq TC_{ENF \bowtie CORF-2}$ (by using $A \setminus (B \cup C) \subseteq (A \setminus B) \subseteq (A \cup D) \setminus B$).

  (b) Some points in TC_{ENF} cannot satisfy $DC_{ENF \bowtie CORF-2}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. For example, in TC_{ENF}, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_2,\bar{j}_2}(S) \subseteq FP(S)$. By definition of $NFP_{i_2,\bar{j}_2}(S)$, $p_{i_2,\bar{j}_2} = 1$ on $\vec{t}$. Note that, $\vec{t}$ does not satisfy $DC_{ENF \bowtie CORF-2}$ because $\vec{t}$ is a false point of S and $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0 + 1 = 1 \neq 0$ on $\vec{t}$.

Hence, only some, but not all, points satisfying DC_{ENF} can satisfy $DC_{ENF \bowtie CORF-2}$.

(B) Relationship between DC_{CORF} and $DC_{ENF \bowtie CORF-2}$: The relationship R6 holds because:

$$
\begin{aligned}
& DC_{CORF} \cap DC_{ENF \ltimes CORF-2} \\
= \; & \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \cap \Bigg(\Big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big) \\
& \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \Bigg) \\
= \; & \Bigg(\{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \cap \Big((\bigcup_{i=i_1}^{h_1} TP_i(S)) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)) \Big) \Bigg) \\
& \cup \Big(\{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \cap \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \Big) \\
= \; & \emptyset.
\end{aligned}
$$

Hence, all points satisfying DC_{CORF} do not satisfy $DC_{ENF \ltimes CORF-2}$.

5.1.6 TNF and TNF

In this section, we analyse the relationship between the double fault involving two TNFs. Since there are two TNFs, we use TNF1 and TNF2 to identify the two different TNFs in double fault TNF1 $\ltimes$ TNF2 for ease of understanding.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the i_1-th term, p_{i_1}, in S is wrongly negated, the corresponding detection condition, denoted by DC_{TNF1}, is "any point in $UTP_{i_1}(S)$ or any point in $FP(S)$". We use TC_{TNF1} to denote the set of all points that satisfy DC_{TNF1}, that is $TC_{TNF1} = UTP_{i_1}(S) \cup FP(S)$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly negated, the corresponding detection condition, denoted by DC_{TNF2}, is "any point in $UTP_{i_2}(S)$ or any point in $FP(S)$". We use TC_{TNF2} to denote the set of all points that satisfy DC_{TNF2}, that is $TC_{TNF2} = UTP_{i_2}(S) \cup FP(S)$.

The double-fault expression is equivalent to Expression (13) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TNF1 \ltimes TNF2}$, is "any point in $\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use $TC_{TNF1 \ltimes TNF2}$ to denote the set of all points that satisfy $DC_{TNF1 \ltimes TNF2}$, that is $TC_{TNF1 \ltimes TNF2} = \Big(\big(TP_{i_1}(S) \cap TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big) \Big) \cup FP(S)$.

(A) Relationship between DC_{TNF1} and $DC_{TNF1 \ltimes TNF2}$: The relationship R2 holds because of the following reasons:

   (a) Some points in TC_{TNF1} can satisfy $DC_{TNF1 \ltimes TNF2}$. For example, in TC_{TNF1}, false points of S (that is, points in $FP(S)$) satisfy $DC_{TNF1 \ltimes TNF2}$.

   (b) Some points in TC_{TNF1} cannot satisfy $DC_{TNF1 \ltimes TNF2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{TNF1}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_1}(S) = \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big)$. Note that, $\vec{t}$ does not satisfy $DC_{TNF1 \ltimes TNF2}$ because $\vec{t} \notin FP(S)$, $\vec{t} \notin TP_{i_2}(S)$ and $i_2 \neq i_1$.

   Hence, only some, but not all, points satisfying DC_{TNF1} can satisfy $DC_{TNF1 \ltimes TNF2}$.

(B) Relationship between DC_{TNF2} and $DC_{TNF1 \ltimes TNF2}$: The relationship R5 holds because of the following reasons:

   (a) Some points in TC_{TNF2} can satisfy $DC_{TNF1 \ltimes TNF2}$. For example, in TC_{TNF2}, false points of S (that is, points in $FP(S)$) satisfy $DC_{TNF1 \ltimes TNF2}$.

   (b) Some points in TC_{TNF2} cannot satisfy $DC_{TNF1 \ltimes TNF2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{TNF2}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_2}(S) = \big(TP_{i_2}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_2}}^{m} TP_i(S)\big)$. Note that, $\vec{t}$ does not satisfy $DC_{TNF1 \ltimes TNF2}$ because $\vec{t} \notin FP(S)$, $\vec{t} \notin TP_{i_1}(S)$ and $i_1 \neq i_2$.

   Hence, only some, but not all, points satisfying DC_{TNF2} can satisfy $DC_{TNF1 \ltimes TNF2}$.

5.1.7 TNF and TOF

In this section, we analyse the relationship between the double fault TNF⋈TOF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the i_1-th term, p_{i_1}, in S is wrongly negated, the corresponding detection condition, denoted by DC_{TNF}, is "any point in $UTP_{i_1}(S)$ or any point in $FP(S)$". We use TC_{TNF} to denote the set of all points that satisfy DC_{TNF}, that is $TC_{TNF} = UTP_{i_1}(S) \cup FP(S)$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly omitted, the corresponding detection condition, denoted by DC_{TOF}, is "any point in $UTP_{i_2}(S)$". We use TC_{TOF} to denote the set of all points that satisfy DC_{TOF}, that is $TC_{TOF} = UTP_{i_2}(S)$.

The double-fault expression is equivalent to Expression (14) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TNF \bowtie TOF}$, is "any point in $\big(TP_{i_1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$ or any point in $FP(S)$". We use $TC_{TNF \bowtie TOF}$ to denote the set of all points that satisfy $DC_{TNF \bowtie TOF}$, that is $DC_{TNF \bowtie TOF} = \Big(TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)\Big) \cup FP(S)$.

(A) Relationship between DC_{TNF} and $DC_{TNF \bowtie TOF}$: The relationship R1 holds because:

$$
\begin{aligned}
& TC_{TNF} \\
= \; & UTP_{i_1}(S) \cup FP(S) \\
= \; & \Big(TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big)\Big) \cup FP(S) \\
= \; & \Bigg(TP_{i_1}(S) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big) \cup TP_{i_2}(S)\Big)\Bigg) \cup FP(S) \\
\subseteq \; & \Big(TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} \big) TP_i(S)\Big) \cup FP(S) \qquad \text{(By using } A \setminus (B \cup C) \subseteq A \setminus B\text{)} \\
= \; & TC_{TNF \bowtie TOF}.
\end{aligned}
$$

Hence, any point satisfying DC_{TNF} satisfies $DC_{TNF \bowtie TOF}$.

(B) Relationship between DC_{TOF} and $DC_{TNF \bowtie TOF}$: The relationship R6 holds because:

$$\begin{aligned}
& TC_{TOF} \cap TC_{TNF \bowtie TOF} \\
= \; & UTP_{i_2}(S) \cap \Big(\big(TP_{i_1}(S) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)) \big) \cup FP(S) \Big) \\
= \; & \Big(UTP_{i_2}(S) \cap \big(TP_{i_1}(S) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)) \big) \Big) \cup \big(UTP_{i_2}(S) \cap FP(S) \big) \\
\subseteq \; & \big(UTP_{i_2}(S) \cap TP_{i_1}(S) \big) \cup \emptyset \qquad \text{(By using } A \setminus B \subseteq A \text{)} \\
= \; & \emptyset \qquad \text{(By definition of } UTP_{i_2}(S) \text{ and } i_1 \neq i_2 \text{).}
\end{aligned}$$

Hence, all points satisfying DC_{TOF} do not satisfy $DC_{TNF \bowtie TOF}$.

5.1.8 TNF and DORF

In this section, we analyse the relationship between the double fault TNF⋈DORF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the i_1-th term, p_{i_1} in S is wrongly negated, the corresponding detection condition, denoted by DC_{TNF}, is "any point in $UTP_{i_1}(S)$ or any point in $FP(S)$". We use TC_{TNF} to denote the set of all points that satisfy DC_{TNF}, that is $TC_{TNF} = UTP_{i_1}(S) \cup FP(S)$.

Similarly, if the subexpression $p_{i_2} + p_{i_2+1}$ in S is wrongly implemented as $p_{i_2} \cdot p_{i_2+1}$, the corresponding detection condition, denoted by DC_{DORF}, is "any point in $UTP_{i_2}(S)$ or any point in $UTP_{i_2+1}(S)$". We use TC_{DORF} to denote the set of all points that satisfy DC_{DORF}, that is $TC_{DORF} = UTP_{i_2}(S) \cup UTP_{i_2+1}(S)$.

For TNF⋈DORF, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (15) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TNF \ltimes DORF-1}$, is "any point in $TP_{i_1}(S) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2, i_2+1}}^{m} TP_i(S) \big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S) \big) \Big)$ or any point in $FP(S)$". We use $TC_{TNF \ltimes DORF-1}$ to denote the set of all points that satisfy $DC_{TNF \ltimes DORF-1}$, that is $TC_{TNF \ltimes DORF-1} = \bigg(TP_{i_1}(S) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2, i_2+1}}^{m} TP_i(S) \big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S) \big) \Big) \bigg) \cup FP(S)$.

(A) Relationship between DC_{TNF} and $DC_{TNF \ltimes DORF-1}$: The relationship R1 holds because:

$$
\begin{aligned}
& TC_{TNF} \\
= \; & UTP_{i_1}(S) \cup FP(S) \\
= \; & \Big(TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S) \big) \Big) \cup FP(S) \\
= \; & \bigg(TP_{i_1}(S) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2, i_2+1}}^{m} TP_i(S) \big) \cup TP_{i_2}(S) \cup TP_{i_2+1}(S) \big) \Big) \bigg) \cup FP(S) \\
\subseteq \; & \bigg(TP_{i_1}(S) \setminus \Big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2, i_2+1}}^{m} TP_i(S) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S) \big) \Big) \bigg) \cup FP(S) \qquad \text{(By using } A \setminus (B \cup C \cup D) \subseteq A \setminus \big(B \cup (C \cap D) \big)\text{)} \\
= \; & TC_{TNF \ltimes DORF-1}.
\end{aligned}
$$

Hence, any point satisfying DC_{TNF} satisfies $DC_{TNF \ltimes DORF-1}$.

(B) Relationship between DC_{DORF} and $DC_{TNF \ltimes DORF-1}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{DORF} \cap TC_{TNF \bowtie DORF-1} \\
= \ & \big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \Bigg(\Bigg(TP_{i_1}(S) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2, i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)\Bigg) \\
& \cup FP(S)\Bigg) \\
= \ & \big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap \Bigg(TP_{i_1}(S) \cup FP(S)\Bigg) \qquad \text{(By using } A \setminus B \subseteq A\text{)} \\
= \ & \Big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap TP_{i_1}(S)\Big) \cup \Big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap FP(S)\Big) \\
\subseteq \ & \Big(\big(UTP_{i_2}(S) \cup UTP_{i_2+1}(S)\big) \cap TP_{i_1}(S)\Big) \cup \emptyset \\
= \ & \big(UTP_{i_2}(S) \cap TP_{i_1}(S)\big) \cup \big(UTP_{i_2+1}(S) \cap TP_{i_1}(S)\big) \\
= \ & \emptyset \qquad \text{(By definitions of } UTP_{i_2}(S) \text{ and } UTP_{i_2+1}(S)\text{, and } i_1 < i_2\text{).}
\end{aligned}
$$

Hence, all points satisfying DC_{DORF} do not satisfy $DC_{TNF \bowtie DORF-1}$.

Case 2 The double-fault expression is equivalent to Expression (16) in Table 1 where $i_1 = i_2$. The detection condition, denoted by $DC_{TNF \bowtie DORF-2}$, is "any point in $TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)$". We use $TC_{TNF \bowtie DORF-2}$ to denote the set of points that satisfy $DC_{TNF \bowtie DORF-2}$, that is $TC_{TNF \bowtie DORF-2} = TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)$.

(A) Relationship between DC_{TNF} and $DC_{TNF \bowtie DORF-2}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{TNF} can satisfy $DC_{TNF \bowtie DORF-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. For example, in TC_{TNF}, we can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_1}(S) = \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big)$ and $\vec{t}$ satisfies $DC_{TNF \bowtie DORF-2}$ because $\vec{t} \in \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big) \subseteq TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big) = TC_{TNF \bowtie DORF-2}$ (by using $A \setminus (B \cup C) \subseteq (A \setminus B)$).

(b) Some points in TC_{TNF} cannot satisfy $DC_{TNF \bowtie DORF-2}$. For example, in TC_{TNF}, false points of S (that is, points in $FP(S)$) do not satisfy $DC_{TNF \bowtie DORF-2}$.

Hence, only some, but not all, points satisfying DC_{TNF} can satisfy $DC_{TNF \ltimes DORF-2}$.

(B) Relationship between DC_{DORF} and $DC_{TNF \ltimes DORF-2}$: The relationship R5 holds because of the following reasons:

(a) Some points in TC_{DORF} can satisfy $DC_{TNF \ltimes DORF-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. Given that $i_1 = i_2$, we can find a point $\vec{t}$ in TC_{DORF} such that $\vec{t} \in UTP_{i_2}(S) = UTP_{i_1}(S) = \big(TP_{i_1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big)$ and $\vec{t}$ satisfies $DC_{TNF \ltimes DORF-2}$.

(b) Some points in TC_{DORF} cannot satisfy $DC_{TNF \ltimes DORF-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. Given that $i_1 = i_2$, we can find a point $\vec{t}$ in TC_{DORF} such that $\vec{t} \in UTP_{i_2+1}(S) = UTP_{i_1+1}(S) = \big(TP_{i_1+1}(S) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1+1}}^{m} TP_i(S)\big)$ and $\vec{t}$ does not satisfy $DC_{TNF \ltimes DORF-2}$ because $\vec{t} \notin TP_{i_1}(S)$.

Hence, only some, but not all, points satisfying DC_{DORF} can satisfy $DC_{TNF \ltimes DORF-2}$.

5.1.9 TNF and CORF

In this section, we analyse the relationship between the double fault TNF⋉CORF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the i_1-th term, p_{i_1} in S is wrongly negated, the corresponding detection condition, denoted by DC_{TNF}, is "any point in $UTP_{i_1}(S)$ or any point in $FP(S)$". We use TC_{TNF} to denote the set of all points that satisfy DC_{TNF}, that is $TC_{TNF} = UTP_{i_1}(S) \cup FP(S)$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$, the corresponding detection condition, denoted by DC_{CORF}, is "any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$". We use TC_{CORF} to denote the set of all points that satisfy DC_{CORF}, that is $TC_{CORF} = \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$.

For TNF$\bowtie$CORF, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (17) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TNF \bowtie CORF-1}$, is "any point in $UTP_{i_1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ or any point in $FP(S)$". We use $TC_{TNF \bowtie CORF-1}$ to denote the set of all points that satisfy $DC_{TNF \bowtie CORF-1}$, that is $TC_{TNF \bowtie CORF-1} = \{\vec{t} \in UTP_{i_1}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup FP(S)$.

(A) Relationship between DC_{TNF} and $DC_{TNF \bowtie CORF-1}$:

First, we observe that $TC_{TNF \bowtie CORF-1} = \{\vec{t} \in UTP_{i_1}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup FP(S) \subseteq (UTP_{i_1}(S) \cup FP(S)) = TC_{TNF}$. Hence the relationship between DC_{TNF} and $DC_{TNF \bowtie CORF-1}$ depends on whether $TC_{TNF \bowtie CORF-1} = TC_{TNF}$.

If $TC_{TNF \bowtie CORF-1}$ and TC_{TNF} are not equal, R2 holds. That means, some points in TC_{TNF} do not satisfy $DC_{TNF \bowtie CORF-1}$. Here is an example. Let $S = abc + \bar{a}\bar{b}d + c\bar{d}$. If the first term abc is negated and the second term is wrongly split as $\bar{a}\bar{b} + d$, $I_{TNF} = \overline{abc} + \bar{a}\bar{b}d + c\bar{d}$ and $I_{TNF \bowtie CORF-1} = \overline{abc} + \bar{a}\bar{b} + d + c\bar{d}$. Now, the point $1111 \in UTP_1(S) \subseteq TC_{TNF}$ can distinguish S from I_{TNF} because S and I_{TNF} evaluate to 1 and 0 on this point, respectively. However, it cannot be used to distinguish S from $I_{TNF \bowtie CORF-1}$ because both S and $I_{TNF \bowtie CORF-1}$ evaluate to 1 on 1111.

On the other hand, if $TC_{TNF \bowtie CORF-1} = TC_{TNF}$, R1 holds. That means, $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ evaluate to 0 on all points in $UTP_{i_1}(S)$. Under such special circumstances, all test cases that can detect $TNF \bowtie CORF-1$ can also be used to detect TNF. Here is an example. Let $S = ab + cd + ac + bd$. Note that $UTP_1(S) = \{1100\}$. If the first term ab is negated and the second term is wrongly split as $c + d$, $I_{TNF} = \overline{ab} + cd + ac + bd$ and $I_{TNF \bowtie CORF-1} = \overline{ab} + c + d + ac + bd$. Now, $TC_{TNF} = UTP_1(S) \cup FP(S) = \{1100\} \cup FP(S)$, and $TC_{TNF \bowtie CORF-1} = \{\vec{t} \in UTP_1(S) : c + d = 0\} \cup FP(S) = \{1100\} \cup FP(S) = TC_{TNF}$ because $c + d = 0$ on all points of $UTP_1(S) = \{1100\}$.

(B) Relationship between DC_{CORF} and $DC_{TNF \bowtie CORF-1}$: The relationship R4 holds because:

$$
\begin{aligned}
& TC_{CORF} \\
= \; & \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \\
\subseteq \; & FP(S) \\
\subseteq \; & \{\vec{t} \in UTP_{i_1}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup FP(S) \\
= \; & TC_{TNF \ltimes CORF-1}.
\end{aligned}
$$

Hence, any point satisfying DC_{CORF} satisfies $DC_{TNF \ltimes CORF-1}$.

Case 2 The double-fault expression is equivalent to Expression (18) in Table 1 where $i_1 = i_2$. The detection condition, denoted by $DC_{TNF \ltimes CORF-2}$, is "any point in $UTP_{i_2}(S)$ or any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$". We use $TC_{TNF \ltimes CORF-2}$ to denote the set of all points that satisfy $DC_{TNF \ltimes CORF-2}$, that is $TC_{TNF \ltimes CORF-2} = UTP_{i_2}(S) \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\}$.

(A) Relationship between DC_{TNF} and $DC_{TNF \ltimes CORF-2}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{TNF} can satisfy $DC_{TNF \ltimes CORF-2}$. Since S is in IDNF, $UTP_i(S) \neq \emptyset$ for all i. Given that $i_1 = i_2$, we can find a point $\vec{t}$ in TC_{TNF} such that $\vec{t} \in UTP_{i_1}(S) = UTP_{i_2}(S)$ and $\vec{t}$ satisfies $DC_{TNF \ltimes CORF-2}$.

(b) Some points in TC_{TNF} cannot satisfy $DC_{TNF \ltimes CORF-2}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. For example, in TC_{TNF}, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_2,\bar{j}_2}(S) \subseteq FP(S)$. By definition of $NFP_{i_2,\bar{j}_2}(S)$, $p_{i_2,\bar{j}_2} = 1$ on $\vec{t}$. Note that, $\vec{t}$ does not satisfy $DC_{TNF \ltimes CORF-2}$ because $\vec{t}$ is a false point of S and $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0 + 1 = 1 \neq 0$ on $\vec{t}$.

Hence, only some, but not all, points satisfying DC_{TNF} can satisfy $DC_{TNF \ltimes CORF-2}$.

(B) Relationship between DC_{CORF} and $DC_{TNF \ltimes CORF-2}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{CORF} \cap TC_{TNF \bowtie CORF-2} \\
= \;& \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \cap \big(UTP_{i_1}(S) \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\}\big) \\
= \;& \big(\{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \cap UTP_{i_1}(S)\big) \\
& \cup \big(\{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \cap \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \\
= \;& \emptyset \quad \text{(because } i_1 = i_2\text{)}.
\end{aligned}
$$

Hence, all points satisfying DC_{CORF} do not satisfy $DC_{TNF \bowtie CORF-2}$.

5.1.10 TOF and TOF

In this section, we analyse the relationship between the double fault involving two TOFs. Since there are two TOFs, we use TOF1 and TOF2 to identify the two different TOFs in double fault TOF1 ⋈ TOF2 for ease of understanding.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the i_1-th term, p_{i_1}, in S is wrongly omitted, the corresponding detection condition, denoted by DC_{TOF1}, is "any point in $UTP_{i_1}(S)$". We use TC_{TOF1} to denote the set of all points that satisfy DC_{TOF1}, that is $TC_{TOF1} = UTP_{i_1}(S)$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly omitted, the corresponding detection condition, denoted by DC_{TOF2}, is "any point in $UTP_{i_2}(S)$". We use TC_{TOF2} to denote the set of all points that satisfy DC_{TOF2}, that is $TC_{TOF2} = UTP_{i_2}(S)$.

The double-fault expression is equivalent to Expression (19) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TOF1 \bowtie TOF2}$, is "any point in $\big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$". We use $TC_{TOF1 \bowtie TOF2}$ to denote the set of all points that satisfy $DC_{TOF1 \bowtie TOF2}$, that is $TC_{TOF1 \bowtie TOF2} = \big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$.

(A) Relationship between DC_{TOF1} and $DC_{TOF1 \bowtie TOF2}$: The relationship R1 holds because:

$$
\begin{aligned}
& DC_{TOF1} \\
= \; & UTP_{i_1}(S) \\
= \; & TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S) \big) \\
= \; & TP_{i_1}(S) \setminus \Big(\big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big) \cup TP_{i_2}(S) \Big) \\
\subseteq \; & TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big) \qquad \text{(By using } A \setminus (B \cup C) \subseteq A \setminus B\text{)} \\
\subseteq \; & \big(TP_{i_1}(S) \cup TP_{i_2}(S) \big) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \qquad \text{(By using } A \setminus B \subseteq (A \cup C) \setminus B\text{)} \\
= \; & TC_{TOF1 \ltimes TOF2}.
\end{aligned}
$$

Hence, any point satisfying DC_{TOF1} satisfies $DC_{TOF1 \ltimes TOF2}$.

(B) Relationship between DC_{TOF2} and $DC_{TOF1 \ltimes TOF2}$: The relationship R4 holds because:

$$
\begin{aligned}
& DC_{TOF2} \\
= \; & UTP_{i_2}(S) \\
= \; & TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_2}}^{m} TP_i(S) \big) \\
= \; & TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \cup TP_{i_1}(S) \big) \\
\subseteq \; & TP_{i_2}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big) \qquad \text{(By using } A \setminus (B \cup C) \subseteq A \setminus B\text{)} \\
\subseteq \; & \big(TP_{i_1}(S) \cup TP_{i_2}(S) \big) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \qquad \text{(By using } A \setminus B \subseteq (A \cup C) \setminus B\text{)} \\
= \; & TC_{TOF1 \ltimes TOF2}.
\end{aligned}
$$

Hence, any point satisfying DC_{TOF2} satisfies $DC_{TOF1 \ltimes TOF2}$.

5.1.11 TOF and DORF

In this section, we analyse the relationship between the double fault TOF⋈DORF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the i_1-th term, p_{i_1} in S is wrongly omitted, the corresponding detection condition, denoted by DC_{TOF}, is "any point in $UTP_{i_1}(S)$". We use TC_{TOF} to denote the set of all points that satisfy DC_{TOF}, that is $TC_{TOF} = UTP_{i_1}(S)$.

Similarly, if the subexpression $p_{i_2} + p_{i_2+1}$ in S is wrongly implemented as $p_{i_2} \cdot p_{i_2+1}$, the corresponding detection condition, denoted by DC_{DORF}, is "any point in $UTP_{i_2}(S)$ or any point in $UTP_{i_2+1}(S)$". We use TC_{DORF} to denote the set of all points that satisfy DC_{DORF}, that is $TC_{DORF} = UTP_{i_2}(S) \cup UTP_{i_2+1}(S)$.

The double-fault expression is equivalent to Expression (20) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TOF \bowtie DORF}$, is "any point in $\Big(\big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big) \Big)$ $\bigcup \Big(\big(TP_{i_1}(S) \cup TP_{i_2+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2+1}}^{m} TP_i(S)\big) \Big)$".

(A) Relationship between DC_{TOF} and $DC_{TOF \bowtie DORF}$: The relationship R1 holds because:

$$
\begin{aligned}
& TC_{TOF} \\
= \; & UTP_{i_1}(S) \\
= \; & TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S) \big) \\
= \; & TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \cup TP_{i_2}(S) \big) \\
\subseteq \; & TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big) \qquad \text{(By using } A \setminus (B \cup C) \subseteq A \setminus B\text{)} \\
\subseteq \; & \big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \big) \qquad \text{(By using } (A \setminus B) \subseteq (A \cup C) \setminus B\text{)} \\
\subseteq \; & \Big(\big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S) \Big) \cup \Big(\big(TP_{i_1}(S) \cup TP_{i_2+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2+1}}^{m} TP_i(S) \big) \Big) \\
= \; & TC_{TOF \ltimes DORF}.
\end{aligned}
$$

Hence, any point satisfying DC_{TOF} satisfies $DC_{TOF \ltimes DORF}$.

(B) Relationship between DC_{DORF} and $DC_{TOF \ltimes DORF}$: The relationship R4 holds because:

$$
\begin{aligned}
& DC_{DORF} \\
= \; & UTP_{i_2}(S) \cup UTP_{i_2+1}(S) \\
= \; & \Big(TP_{i_2}(S) \setminus (\bigcup_{\substack{i=1 \\ i\neq i_2}}^{m} TP_i(S))\Big) \cup \Big(TP_{i_2+1}(S) \setminus (\bigcup_{\substack{i=1 \\ i\neq i_2+1}}^{m} TP_i(S))\Big) \\
= \; & \Big(TP_{i_2}(S) \setminus \Big((\bigcup_{\substack{i=1 \\ i\neq i_1,i_2}}^{m} TP_i(S)) \cup TP_{i_1}(S)\Big)\Big) \cup \Big(TP_{i_2+1}(S) \setminus \Big((\bigcup_{\substack{i=1 \\ i\neq i_1,i_2+1}}^{m} TP_i(S)) \cup TP_{i_1}(S)\Big)\Big) \\
\subseteq \; & \Big(TP_{i_2}(S) \setminus (\bigcup_{\substack{i=1 \\ i\neq i_1,i_2}}^{m} TP_i(S))\Big) \cup \Big(TP_{i_2+1}(S) \setminus (\bigcup_{\substack{i=1 \\ i\neq i_1,i_2+1}}^{m} TP_i(S))\Big) \\
& \quad \text{(By using } A \setminus (B \cup C) \subseteq A \setminus B\text{)} \\
\subseteq \; & \Big(\big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus (\bigcup_{\substack{i=1 \\ i\neq i_1,i_2}}^{m} TP_i(S))\Big) \cup \Big(\big(TP_{i_1}(S) \cup TP_{i_2+1}(S)\big) \setminus (\bigcup_{\substack{i=1 \\ i\neq i_1,i_2+1}}^{m} TP_i(S))\Big) \\
& \quad \text{(By using } (A \setminus B) \subseteq (C \cup A) \setminus B\text{)} \\
= \; & TC_{TOF \bowtie DORF}.
\end{aligned}
$$

Hence, any point satisfying DC_{DORF} satisfies $DC_{TOF \bowtie DORF}$.

5.1.12 TOF and CORF

In this section, we analyse the relationship between the double fault TOF⋈CORF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the i_1-th term, p_{i_1} in S is wrongly omitted, the corresponding detection condition, denoted by DC_{TOF}, is "any point in $UTP_{i_1}(S)$". We use TC_{TOF} to denote the set of all points that satisfy DC_{TOF}, that is $TC_{TOF} = UTP_{i_1}(S)$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$, the corresponding detection condition, denoted by DC_{CORF}, is "any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ = 1". We use TC_{CORF} to denote the set of all points that satisfy

DC_{CORF}, that is $TC_{CORF} = \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$.

The double-fault expression is equivalent to Expression (21) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{TOF \ltimes CORF}$, is "any point in $UTP_{i_1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ or any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$". We use $TC_{TOF \ltimes CORF}$ to denote the set of all points that satisfy $DC_{TOF \ltimes CORF}$, that is $TC_{TOF \ltimes CORF} = \{\vec{t} \in UTP_{i_1}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$.

(A) Relationship between DC_{TOF} and $DC_{TOF \ltimes CORF-1}$:

Note that, $TC_{TOF} = UTP_{i_1}(S)$ and $TC_{TOF \ltimes CORF} = \{\vec{t} \in UTP_{i_1}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$.

Hence, the relationship DC_{TOF} and $DC_{TOF \ltimes CORF-1}$ depends on whether there are points in $UTP_{i_1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$. There are three subcases.

(a) Some points in $UTP_{i_1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ and some points in $UTP_{i_1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$. Then, R2 holds. Here is an example. Let $S = ab + cd + ef$. If the first term ab is omitted and the second term is wrongly splited as $c + d$, $I_{TOF} = cd + ef$ and $I_{TOF \ltimes CORF-1} = c + d + ef$. Note that, the unique true point 110010 in $UTP_1(S) = TC_{TOF}$ can detect $I_{TOF \ltimes CORF-1}$ because $c + d = 0$ on this point. However, another unique true point 110110 in $UTP_1(S) = TC_{TOF}$ does not satisfy $DC_{TOF \ltimes CORF-1}$ because $c + d = 1$ on this point.

(b) All points in $UTP_{i_1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$. Then, R1 holds. Here is an example. Let $S = ab + cd + ac + bd$. If the first term ab is omitted and the second term is wrongly splited as $c + d$, $I_{TOF} = cd + ac + bd$ and $I_{TOF \ltimes CORF-1} = c + d + ac + bd$. Note that $TC_{TOF} = UTP_1(S) = \{1100\}$, and $TC_{TOF \ltimes CORF-1} = \{\vec{t} \in UTP_1(S) : c + d = 0\} = \{1100\}$ because $c + d = 0$ on 1100. Hence, any point that can detect I_{TOF} can also detect $I_{TOF \ltimes CORF-1}$.

(c) All points in $UTP_{i_1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$. Then, R3 holds. Let $S = abc + \bar{a}\bar{b}d + c\bar{d}$. If the first term abc is omitted and the second term is wrongly splited as $\bar{a}\bar{b} + d$,

$I_{TOF} = \bar{a}\bar{b}d + c\bar{d}$ and $I_{TOF \bowtie CORF-1} = \bar{a}\bar{b} + d + c\bar{d}$. Note that, $TC_{TOF} = UTP_1(S) = \{1111\}$. The point 1111 can distinguish S from I_{TOF} because S and I_{TOF} evaluate to 1 and 0 on this point, respectively. However, it cannot be used to distinguish S from $I_{TOF \bowtie CORF-1}$ because both S and $I_{TOF \bowtie CORF-1}$ evaluate to 1 on 1111.

(B) Relationship between DC_{CORF} and $DC_{TOF \bowtie CORF}$: The relationship R4 holds becuase:

$$
\begin{aligned}
& TC_{CORF} \\
= \;& \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \\
\subseteq \;& \{\vec{t} \in UTP_{i_1}(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \\
= \;& TC_{TOF \bowtie CORF}.
\end{aligned}
$$

Hence, any point satisfying DC_{CORF} satisfies $DC_{TOF \bowtie CORF-1}$.

5.1.13 DORF and DORF

In this section, we analyse the relationship between the double fault involving two DORFs. Since there are two DORFs, we use DORF1 and DORF2 to identify the two different DORFs in double fault DORF1 $\bowtie$ DORF2 for ease of understanding.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + p_{i_1+1}$ in S is wrongly implemented as $p_{i_1} \cdot p_{i_1+1}$, the corresponding detection condition, denoted by DC_{DORF1}, is "any point in $UTP_{i_1}(S)$ or any point in $UTP_{i_1+1}(S)$". We use TC_{DORF1} to denote the set of all points that satisfy DC_{DORF1}, that is $TC_{DORF1} = UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$.

Similarly, if the subexpression $p_{i_2} + p_{i_2+1}$ in S is wrongly implemented as $p_{i_2} \cdot p_{i_2+1}$, the corresponding detection condition, denoted by DC_{DORF1}, is "any point in $UTP_{i_2}(S)$ or any point in $UTP_{i_2+1}(S)$". We use TC_{DORF2} to denote the set of all points that satisfy DC_{DORF2}, that is $TC_{DORF2} = UTP_{i_2}(S) \cup UTP_{i_2+1}(S)$.

For DORF1 $\bowtie$ DORF2, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (22) in Table 1 where $i_1 + 1 < i_2$. The detection condition, denoted by $DC_{DORF1 \ltimes DORF2-1}$, is "any point in $\big(\bigcup_{i=i_1,i_1+1,i_2,i_2+1} TP_i(S)\big) \setminus \Big(\big(\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$". We use $TC_{DORF1 \ltimes DORF2-1}$ to denote the set of all points that satisfy $DC_{DORF1 \ltimes DORF2-1}$, that is $TC_{DORF1 \ltimes DORF2-1} = \big(\bigcup_{i=i_1,i_1+1,i_2,i_2+1} TP_i(S)\big) \setminus \Big(\big(\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$.

(A) Relationship between DC_{DORF1} and $DC_{DORF1 \ltimes DORF2-1}$: The relationship R1 holds because:

$$
\begin{aligned}
& TC_{DORF1} \\
= \;& UTP_{i_1}(S) \cup UTP_{i_1+1}(S) \\
= \;& \Big(TP_{i_1}(S) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1}}^{m} TP_i(S))\Big) \cup \Big(TP_{i_1+1}(S) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1+1}}^{m} TP_i(S))\Big) \\
= \;& \Big(TP_{i_1}(S) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)) \cup TP_{i_1+1}(S) \cup TP_{i_2}(S) \cup TP_{i_2+1}(S)\Big)\Big) \\
& \cup \Big(TP_{i_1+1}(S) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)) \cup TP_{i_1}(S) \cup TP_{i_2}(S) \cup TP_{i_2+1}(S)\Big)\Big) \\
\subseteq \;& \Big(TP_{i_1}(S) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)\Big) \\
& \cup \Big(TP_{i_1+1}(S) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)\Big)
\end{aligned}
$$

(By using $A \setminus (B \cup C \cup D \cup E) \subseteq A \setminus \big(B \cup (A \cap C) \cup (D \cap E)\big)$ because $A \cap C \subseteq C$ and $D \cap E \subseteq D \cup E$)

$$
\begin{aligned}
= \;& \big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big) \\
\subseteq \;& (\bigcup_{i=i_1,i_1+1,i_2,i_2+1} TP_i(S)) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)
\end{aligned}
$$

(By using $(A \setminus B) \subseteq (A \cup C) \setminus B$)

$$
= \; TC_{DORF1 \ltimes DORF2-1}
$$

Hence, any point satisfying DC_{DORF1} satisfies $DC_{DORF1 \ltimes DORF2-1}$.

(B) Relationship between DC_{DORF2} and $DC_{DORF2 \ltimes DORF2-1}$: The relationship R4 holds because:

$$
\begin{aligned}
& TC_{DORF2} \\
= \;& UTP_{i_2}(S) \cup UTP_{i_2+1}(S) \\
\subseteq \;& \Big(\bigcup_{i=i_1,i_1+1,i_2,i_2+1} TP_i(S)\Big) \setminus \Big(\Big(\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)\Big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap \\
& TP_{i_2+1}(S)\big)\Big) \\
= \;& TC_{DORF1 \ltimes DORF2-1} \quad \text{(By using a similar argument as in (A) above)}
\end{aligned}
$$

Hence, any point satisfying DC_{DORF2} satisfies $DC_{DORF1 \ltimes DORF2-1}$.

Case 2 The double-fault expression is equivalent to Expression (23) in Table 1 where $i_1 + 1 = i_2$. The detection condition, denoted by $DC_{DORF1 \ltimes DORF2-2}$, is "any point in $\Big(\bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S)\Big) \setminus \Big(\Big(\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_1+2}}^{m} TP_i(S)\Big) \cup \Big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\Big)\Big)$". We use $TC_{DORF1 \ltimes DORF2-2}$ to denote the set of all points that satisfy $DC_{DORF1 \ltimes DORF2-2}$, that is $TC_{DORF1 \ltimes DORF2-2} = \Big(\bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S)\Big) \setminus \Big(\Big(\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_1+2}}^{m} TP_i(S)\Big) \cup \Big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\Big)\Big)$.

(A) Relationship between DC_{DORF1} and $DC_{DORF1 \ltimes DORF2-2}$: The relationship R1 holds because:

$$
\begin{aligned}
& TC_{DORF1} \\
= \ & UTP_{i_1}(S) \cup UTP_{i_1+1}(S) \\
= \ & \Big(TP_{i_1}(S) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1}}^{m} TP_i(S))\Big) \cup \Big(TP_{i_1+1}(S) \setminus (\bigcup_{\substack{i=1\\ i\neq i_1+1}}^{m} TP_i(S))\Big) \\
= \ & \Big(TP_{i_1}(S) \setminus \big((\bigcup_{\substack{i=1\\ i\neq i_1, i_1+1, i_1+2}}^{m} TP_i(S)) \cup TP_{i_1+1}(S) \cup TP_{i_1+2}(S)\big)\Big) \\
& \cup \Big(TP_{i_1+1}(S) \setminus \big((\bigcup_{\substack{i=1\\ i\neq i_1, i_1+1, i_1+2}}^{m} TP_i(S)) \cup TP_{i_1}(S) \cup TP_{i_1+2}(S)\big)\Big) \\
\subseteq \ & \Big(TP_{i_1}(S) \setminus \big((\bigcup_{\substack{i=1\\ i\neq i_1, i_1+1, i_1+2}}^{m} TP_i(S)) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)\big)\Big) \\
& \cup \Big(TP_{i_1+1}(S) \setminus \big((\bigcup_{\substack{i=1\\ i\neq i_1, i_1+1, i_1+2}}^{m} TP_i(S)) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)\big)\Big) \\
& \qquad \text{(By using } A \setminus (B \cup C \cup D) \subseteq A \setminus \big(B \cup (A \cap C \cap D)\big) \text{ because } (A \cap C \cap D) \subseteq C \cup D\text{)} \\
= \ & \big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1, i_1+1, i_1+2}}^{m} TP_i(S)) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)\Big) \\
\subseteq \ & (\bigcup_{i=i_1, i_1+1, i_1+2} TP_i(S)) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1, i_1+1, i_1+2}}^{m} TP_i(S)) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)\Big) \\
& \qquad \text{(By using } (A \setminus B) \subseteq (A \cup C) \setminus B\text{)} \\
= \ & TC_{DORF1 \ltimes DORF2-2}
\end{aligned}
$$

Hence, any points satisfying DC_{DORF1} can satisfy $DC_{DORF1 \ltimes DORF2-2}$.

(B) Relationship between DC_{DORF2} and $DC_{DORF1 \ltimes DORF2-2}$: The relationship R4 holds because:

$$
\begin{aligned}
& TC_{DORF2} \\
= \ & UTP_{i_1+1}(S) \cup UTP_{i_1+2}(S) \\
\subseteq \ & (\bigcup_{i=i_1, i_1+1, i_1+2} TP_i(S)) \setminus \Big((\bigcup_{\substack{i=1\\ i\neq i_1, i_1+1, i_1+2}}^{m} TP_i(S)) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)\Big) \\
= \ & TC_{DORF1 \ltimes DORF2-2} \qquad \text{(By using a similar argument as in (A) above)}
\end{aligned}
$$

Hence, any points satisfying DC_{DORF2} can satisfy $DC_{DORF1 \ltimes DORF2-2}$.

5.1.14 DORF and CORF

In this section, we analyse the relationship between the double fault DORF⋈CORF and its corresponding single faults.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the subexpression $p_{i_1} + p_{i_1+1}$ in S is wrongly implemented as $p_{i_1} \cdot p_{i_1+1}$, the corresponding detection condition, denoted by DC_{DORF}, is "any point in $UTP_{i_1}(S)$ or any point in $UTP_{i_1+1}(S)$". We use TC_{DORF} to denote the set of all points that satisfy DC_{DORF}, that is $TC_{DORF} = UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$, the corresponding detection condition, denoted by DC_{CORF}, is "any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ = 1". We use TC_{CORF} to denote the set of all points that satisfy DC_{CORF}, that is $TC_{CORF} = \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$.

For DORF⋈CORF, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (24) in Table 1 where $i_1 < i_2 - 1$. The detection condition, denoted by $DC_{DORF\bowtie CORF-1}$, is "any point in $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ or any point in $FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ = 1". We use $TC_{DORF\bowtie CORF-1}$ to denote the set of all points that satisfy $DC_{DORF\bowtie CORF-1}$, that is $TC_{DORF\bowtie CORF-1}$ = $\{\vec{t} \in \left(UTP_{i_1}(S) \cup UTP_{i_1+1}(S)\right) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$.

(A) Relationship between DC_{DORF} and $DC_{DORF\bowtie CORF-1}$: We observe that $TC_{DORF} = UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ and $TC_{DORF\bowtie CORF-1} = \{\vec{t} \in \left(UTP_{i_1}(S) \cup UTP_{i_1+1}(S)\right) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$. Hence, the relationship between DC_{DORF} and $DC_{DORF\bowtie CORF-1}$ depends on whether there are points in $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$. There are three cases:

(a) Some points in $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$ and some points in $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$. Then, R2 holds. Here is an example. Let $S = ab + cd + ef$. If the subexpression $ab + cd$ is implemented as $abcd$ and the third term ef is wrongly split as $e + f$, $I_{DORF} = abcd + ef$ and $I_{DORF \ltimes CORF-1}$ = $abcd + e + f$. Note that, the unique true point 110000 in $UTP_1(S) \subseteq TC_{DORF}$ can detect $I_{DORF \ltimes CORF-1}$ because $e + f$ evaluates to 0 on this point in $UTP_1(S)$. However, another unique true point 110110 in $UTP_1(S) \subseteq TC_{DORF}$ is such that $I_{DORF \ltimes CORF-1}(110110) = 1$, and hence cannot be used to detect $I_{DORF \ltimes CORF-1}$.

(b) All points in $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0$. Then, R1 holds. Here is an example. Let $S = ab + cd + ef + ae + bf + ce + df$. If the subexpression $ab + cd$ is wrongly implemented as $abcd$ and the third term is wrongly splited as $e + f$, $I_{DORF} = abcd + ef + ae + bf + ce + df$ and $I_{DORF \ltimes CORF-1} = abcd + e + f + ae + bf + ce + df$. Note that $TC_{DORF} = UTP_1(S) \cup UTP_2(S)$ = $\{110000, 110100, 111000, 001100, 011100, 101100\}$, and $TC_{DORF \ltimes CORF-1} = \{\vec{t} \in UTP_1(S) \cup UTP_2(S) : e + f = 0\} = UTP_1(S) \cup UTP_2$ because $e + f = 0$ on all points in TC_{DORF} . Hence, $TC_{DORF} = TC_{DORF \ltimes CORF-1}$.

(c) All points $UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ are such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$. Then, R3 holds. Here is an example. Let $S = ab + cd + ac$. If the subexpression $ab + cd$ is wrongly implemented as $abcd$ and the third term is wrongly splited as $a + c$, $I_{DORF} = abcd + ac$ and $I_{DORF \ltimes CORF-1} = abcd + a + c$. Note that $TC_{DORF} = UTP_1(S) \cup UTP_2(S)$ = $\{1100, 1101, 1110, 0011, 0111, 1011\}$, and $TC_{DORF \ltimes CORF-1} = \{\vec{t} \in UTP_1(S) \cup UTP_2(S) : a + c = 0\} = \emptyset$ because $a + c = 1$ on all points in $UTP_1(S) \cup UTP_2(S)$. Hence, all points in TC_{DORF} cannot be used to detect $I_{DORF \ltimes CORF-1}$.

(B) Relationship between DC_{CORF} and $DC_{DORF \ltimes CORF-1}$: The relationship R4 holds because:

$$
\begin{aligned}
& DC_{CORF} \\
= \; & \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \\
\subseteq \; & \{\vec{t} \in \left(UTP_{i_1}(S) \cup UTP_{i_1+1}(S)\right) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + \\
& p_{i_2,j_2+1,k_{i_2}} = 1\} \\
= \; & TC_{DORF \ltimes CORF-1}.
\end{aligned}
$$

Hence any point satisfying DC_{CORF} satisfies $DC_{DORF \ltimes CORF-1}$.

Case 2 The double-fault expression is equivalent to Expression (25) in Table 1 where $i_1 = i_2$. The detection condition, denoted by $DC_{DORF \ltimes CORF-2}$, is "any point in $UTP_{i_1+1}(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0$ or any point in $FP(S)$ such that $p_{i_1,1,j_1} = 1$". We use $TC_{DORF \ltimes CORF-2}$ to denote the set of all points that satisfy $DC_{DORF \ltimes CORF-2}$, that is $TC_{DORF \ltimes CORF-2} = \{\vec{t} \in UTP_{i_1+1}(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 1\}$.

(A) Relationship between DC_{DORF} and $DC_{DORF \ltimes CORF-2}$: $TC_{DORF} = UTP_{i_1}(S) \cup UTP_{i_1+1}(S)$ and $TC_{DORF \ltimes CORF-2} = \{\vec{t} \in UTP_{i_1+1}(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 1\}$.

Hence, the relationship between DC_{DORF} and $DC_{DORF \ltimes CORF-2}$ depends on whether there are points in UTP_{i_1+1} such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0$. There are two cases:

(a) Some points in $UTP_{i_1+1}(S)$ are such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0$. Then R2 holds because of the following reasons:

(1) Some points in TC_{DORF} can satisfy $DC_{DORF \ltimes CORF-2}$. For example, we can find a point $\vec{t}$ such that $\vec{t} \in \{\vec{t} \in UTP_{i_1+1}(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0\} \subseteq UTP_{i_1+1}(S) \subseteq TC_{DORF}$. Note that, $\vec{t}$ satisfies $DC_{DORF \ltimes CORF-2}$ because $\vec{t} \in UTP_{i_1+1}(S)$ and $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0$.

(2) Some points in TC_{DORF} cannot satisfy $DC_{DORF \ltimes CORF-2}$. We can find a point $\vec{t}$ such that $\vec{t} \in UTP_{i_1}(S) \subseteq TC_{DORF}$. Note that, $\vec{t}$ does not satisfy $DC_{DORF \ltimes CORF-2}$ because $\vec{t} \notin UTP_{i_1+1}(S)$ and $\vec{t} \notin FP(S)$.

(b) All points $\vec{t}$ in $UTP_{i_1+1}(S)$ are such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1$. Then R3 holds. Here is an example, let $S = ab + ac + d$. If the first two terms ab and ac are implemented as $abac$ and the first term is wrongly split as $a + b$, $I_{DORF} = abac + d$ and $I_{DORF \ltimes CORF-2} = a + bac + d$. Note that, $TC_{DORF} = UTP_1(S) \cup UTP_2(S) = \{1100, 1010\}$. The two points 1100 and 1010 can distinguish S from I_{DORF} because S and I_{DORF} evaluate to 1 and 0 on these points, respectively. However, they cannot be used to distinguish S from $I_{DORF \ltimes CORF-2}$ because both S and $I_{DORF \ltimes CORF-2}$ evaluate to 1 on these points.

(B) Relationship between DC_{CORF} and $DC_{DORF \bowtie CORF-2}$: The relationship R5 holds because of the following reasons:

(a) Some points in TC_{CORF} can satisfy $DC_{DORF \bowtie CORF-2}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{k}_{i_1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x^{i_1}_{k_{i_1}} = 0$. Hence $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1 + 0 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{k}_{i_1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \subseteq TC_{CORF}$. Note that, $\vec{t}$ satisfies $DC_{DORF \bowtie CORF-2}$ because $\vec{t}$ is a false point of S and $p_{i_1,1,j_1} = 1$ on $\vec{t}$.

(b) Some points in TC_{CORF} cannot satisfy $DC_{CORF \bowtie ENF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x^{i_1}_1 = 0$. Hence $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \subseteq TC_{CORF}$. Note that, $\vec{t}$ cannot satisfy $DC_{DORF \bowtie CORF-2}$ because $p_{i_1,1,j_1} = 0$.

Hence, only some, but not all, points satisfying DC_{CORF} can satisfy $DC_{DORF \bowtie CORF-2}$.

5.1.15 CORF and CORF

In this section, we analyse the relationship between the double fault involving two CORFs. Since there are two CORFs, we use CORF1 and CORF2 to identify the two different CORFs in double fault CORF1 $\bowtie$ CORF2 for ease of understanding.

Let S be a Boolean expression in IDNF. As discussed in Section 2.2, if the i_1-th term, p_{i_1}, in S is wrongly implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$, the corresponding detection condition, denoted by DC_{CORF1}, is "any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1$". We use TC_{CORF1} to denote the set of all points that satisfy DC_{CORF1}, that is $TC_{CORF1} = \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\}$.

Similarly, if the i_2-th term, p_{i_2}, in S is wrongly implemented as $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ where $p_{i_2} = p_{i_2,1,j_2} \cdot p_{i_2,j_2+1,k_{i_2}}$, the corresponding detection condition, denoted by DC_{CORF2}, is "any point in

$FP(S)$ such that $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ = 1". We use TC_{CORF2} to denote the set of all points that satisfy DC_{CORF2}, that is $TC_{CORF2} = \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$.

For CORF1⋈CORF2, there are two subcases.

Case 1 The double-fault expression is equivalent to Expression (26) in Table 1 where $i_1 < i_2$. The detection condition, denoted by $DC_{CORF1 \bowtie CORF2-1}$, is "any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$". We use $TC_{CORF1 \bowtie CORF2-1}$ to denote the set of all points that satisfy $DC_{CORF1 \bowtie CORF2-1}$, that is $TC_{CORF1 \bowtie CORF2-1} = \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\}$.

(A) Relationship between DC_{CORF1} and $DC_{CORF1 \bowtie CORF2-1}$: The relationship R1 holds because:

$$\begin{aligned} & TC_{CORF1} \\ = \ & \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \\ \subseteq \ & \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \\ = \ & DC_{CORF1 \bowtie CORF2-1}. \end{aligned}$$

Hence, any point satisfying DC_{CORF1} satisfies $DC_{CORF1 \bowtie CORF2-1}$.

(B) Relationship between DC_{CORF2} and $DC_{CORF1 \bowtie CORF2-1}$: The relationship R4 holds because:

$$\begin{aligned} & TC_{CORF2} \\ = \ & \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \\ \subseteq \ & \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \\ = \ & DC_{CORF1 \bowtie CORF2-1} \end{aligned}$$

Hence, any point satisfying DC_{CORF1} satisfies $DC_{CORF1 \bowtie CORF2-1}$.

Case 2 The double-fault expression is equivalent to Expression (27) in Table 1 where $i_1 = i_2$ and $j_1 < j_2$. The detection condition, denoted by $DC_{CORF \bowtie CORF-2}$, is "any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}}$ = 1". We use $TC_{CORF1 \bowtie CORF2-2}$ to denote the set of all points that satisfy $DC_{CORF1 \ltimes CORF2-2}$, that is $TC_{CORF1 \ltimes CORF2-2} = \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1\}$.

(A) Relationship between DC_{CORF} and $DC_{CORF \ltimes CORF-2}$: The relationship R1 holds because:

$$\begin{aligned}
& TC_{CORF1} \\
= \ & \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \\
\subseteq \ & \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1\}
\end{aligned}$$

(Note that, either all literals of $p_{i_1,1,j_1}$ or all literals of $p_{i_1,j_1+1,k_{i_1}}$ evaluate to 1 on these points.)

$$= \ DC_{CORF1 \ltimes CORF2-2}$$

Hence, any point satisfying DC_{CORF} satisfies $DC_{CORF \ltimes CORF-2}$.

(B) Relationship between DC_{CORF} and $DC_{CORF \ltimes CORF-2}$: The relationship R4 holds because:

$$\begin{aligned}
& TC_{CORF2} \\
= \ & \{\vec{t} \in FP(S) : p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1\} \\
= \ & \{\vec{t} \in FP(S) : p_{i_1,1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1\} \qquad (i_1 = i_2) \\
\subseteq \ & \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1\}
\end{aligned}$$

(Note that, either all literals of $p_{i_1,1,j_2}$ or all literals of $p_{i_1,j_2+1,k_{i_1}}$ evaluate to 1 on these points.)

$$= \ DC_{CORF1 \ltimes CORF2-2}$$

Hence, any point satisfying DC_{CORF} satisfies $DC_{CORF \ltimes CORF-2}$.

5.2 Fault Coupling on 4 Remaining Faulty Implementations

As mentioned previously, in double fault with ordering, there are 4 double-fault expressions that do not have their counterparts in the double faults without ordering. In this section, we analyse the relationships between their detection conditions and those of their corresponding single faults.

5.2.1 TOF and DORF

In this section, we analyse the relationship between the double fault TOF⋈DORF and its corresponding single faults with respect to the situation of Expression (53) in Table 1.

Let S be a Boolean expression in IDNF. The double fault expression for TOF and then DORF corresponding to Expression (53) can be derived from the situation where the i_1-th term, p_{i_1}, is omitted first and then the two consecutive terms p_{i_1-1} and p_{i_1+1} are wrongly implemented as $p_{i_1-1} \cdot p_{i_1+1}$.

As discussed in Section 2.2, if the i_1-th term, p_{i_1}, in S is wrongly omitted, the corresponding detection condition, denoted by DC_{TOF}, is "any point in $UTP_{i_1}(S)$". We use TC_{TOF} to denote the set of all points that satisfy DC_{TOF}, that is $TC_{TOF} = UTP_{i_1}(S)$.

Similarly, if the subexpression $p_{i_1-1} + p_{i_1+1}$ in S is wrongly implemented as $p_{i_1-1} \cdot p_{i_1+1}$, the corresponding detection condition, denoted by DC_{DORF}, is "any point in $UTP_{i_1-1}(S)$ or any point in $UTP_{i_1+1}(S)$". We use TC_{DORF} to denote the set of all points that satisfy DC_{DORF}, that is $TC_{DORF} = UTP_{i_1-1}(S) \cup UTP_{i_1+1}(S)$.

Hence, the corresponding detection condition of Expression (53), denoted by $DC_{TOF \bowtie DORF}$, is "any point in $\Big(\big(TP_{i_1-1}(S) \cup TP_{i_1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1-1, i_1}}^{m} TP_i(S)\big)\Big) \cup \Big(\big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)\Big)$". We use $TC_{TOF \bowtie DORF}$ to denote the set of all points that satisfy $DC_{TOF \bowtie DORF}$, that is $TC_{TOF \bowtie DORF} = \Big(\big(TP_{i_1-1}(S) \cup TP_{i_1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1-1, i_1}}^{m} TP_i(S)\big)\Big) \cup \Big(\big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)\Big)$.

(A) Relationship between DC_{TOF} and $DC_{TOF \bowtie DORF}$: The relationship R1 holds because:

$$
\begin{aligned}
& TC_{TOF} \\
= \; & UTP_{i_1}(S) \\
= \; & TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S) \big) \\
= \; & TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S) \cup TP_{i_1+1}(S) \big) \\
\subseteq \; & TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S) \big) \qquad \text{(By using } \big(A \setminus (B \cup C)\big) \subseteq (A \setminus B)\text{)} \\
\subseteq \; & \big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S) \big) \qquad \text{(By using } (A \setminus B) \subseteq (A \cup C) \setminus B\text{)} \\
\subseteq \; & \Big(\big(TP_{i_1-1}(S) \cup TP_{i_1}(S)\big) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1-1, i_1}}^{m} TP_i(S) \Big) \cup \Big(\big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus \bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S) \Big) \\
= \; & TC_{TOF \ltimes DORF} \qquad \text{(By using } B \subseteq A \cup B\text{)}
\end{aligned}
$$

Hence, any point satisfying DC_{TOF} satisfies $DC_{TOF \ltimes DORF}$.

(B) Relationship between DC_{DORF} and $DC_{TOF \ltimes DORF}$: The relationship R4 holds because:

$$
\begin{aligned}
& DC_{DORF} \\
= \ & UTP_{i_1-1}(S) \cup UTP_{i_1+1}(S) \\
= \ & \Big(TP_{i_1-1}(S) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1-1}}^{m} TP_i(S))\Big) \cup \Big(TP_{i_1+1}(S) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1+1}}^{m} TP_i(S))\Big) \\
= \ & \Big(TP_{i_1-1}(S) \ \setminus \ \Big((\bigcup_{\substack{i=1 \\ i \neq i_1-1,i_1}}^{m} TP_i(S))) \ \cup \ TP_{i_1}(S)\Big)\Big) \ \cup \ \Big(TP_{i_1+1}(S) \ \setminus \ \Big((\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1}}^{m} TP_i(S)) \ \cup \\
& TP_{i_1}(S))\Big) \\
\subseteq \ & \Big(TP_{i_1-1}(S) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1-1,i_1}}^{m} TP_i(S))\Big) \cup \Big(TP_{i_1+1}(S) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1}}^{m} TP_i(S))\Big) \\
& \quad \text{(By using } \big(A \setminus (B \cup C)\big) \subseteq (A \setminus B)\text{)} \\
\subseteq \ & \Big(\big(TP_{i_1-1}(S) \cup TP_{i_1}(S)\big) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1-1,i_1}}^{m} TP_i(S))\Big) \cup \Big(\big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus (\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1}}^{m} TP_i(S))\Big) \\
& \quad \text{(By using } (A \setminus B) \subseteq (A \cup C) \setminus B\text{)} \\
= \ & TC_{TOF \bowtie DORF}
\end{aligned}
$$

Hence, any point satisfying DC_{DORF} satisfies $DC_{TOF \bowtie DORF}$.

5.2.2 CORF and ENF

In this section, we analyse the relationship between the double fault CORF⋈ENF and its corresponding single faults.

Let S be a Boolean expression in IDNF. The double fault expression given by Expression (73) can be derived from the situation where the i_1-th term, p_{i_1}, is wrongly implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$ and then the subexpression $p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}$ is wrongly negated.

As discussed in Section 2.2, if the i_1-th term, p_{i_1}, in S is wrongly implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$, the corresponding implementation, denoted by I_{CORF} will be equivalent to $p_1 + \cdots + p_{i_1-1} + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_{h_1} + \cdots + p_m$, and its detection condition, denoted by DC_{CORF}, is "any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ = 1". We use TC_{CORF} to denote the set of all points that satisfy DC_{CORF}, that is $TC_{CORF} = \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\}$.

Furthermore, given I_{CORF}, if the subexpression $p_{i_1,j_1+1,k_{i_1}} + \cdots + p_{h_1}$ is wrongly negated, the corresponding implementation, denoted by $I_{CORF \ltimes ENF}$, will be equivalent to $p_1 + \cdots + p_{i_1-1} + p_{i_1,1,j_1} + \overline{p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_{h_1}} + \cdots + p_m$, and its detection condition *with respect to* I_{CORF}, denoted by DC_{ENF}, is "any point in $\{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_{h_1} = 1$ and $p_1 + \cdots + p_{i_1,1,j_1} + p_{h_1+1} + \cdots + p_m = 0\}$ or any point in $\{\vec{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + \cdots + p_m = 0\}$". We use TC_{ENF} to denote the set of all points that satisfy DC_{ENF}, that is $TC_{ENF} = \big(\{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_{h_1} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + p_{h_1+1} + \cdots + p_m = 0\}\big) \cup \big(\{\vec{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + \cdots + p_m = 0\}\big)$. For ease of comparison, we express the condition in terms of S as follows:

$$\big(\{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_{h_1} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + p_{h_1+1} + \cdots + p_m = 0\}\big) \cup \big(\{\vec{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + \cdots + p_m = 0\}\big)$$

$$\equiv \big(\{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}}(\overline{p_{i_1+1} + \cdots + p_{h_1}}) + p_{i_1+1} + \cdots + p_{h_1} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1}p_{i_1,j_1+1,k_{i_1}} + p_{h_1+1} + \cdots + p_m = 0 \text{ and } p_{i_1,1,j_1} = 0\}\big) \cup \big(\{\vec{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i_1,1,j_1}p_{i_1,j_1+1,k_{i_1}} + \cdots + p_m = 0 \text{ and } p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\}$$ (By making use of $A + B \equiv A\overline{B} + B$)

$$\equiv \big(\{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}}(\overline{p_{i_1+1} + \cdots + p_{h_1}}) = 1 \text{ and } p_1 + \cdots + p_{i_1} + p_{h_1+1} + \cdots + p_m = 0 \text{ and } p_{i_1,1,j_1} = 0\}\big) \cup \big(\{\vec{t} \in \mathbb{B}^n : p_{i_1+1} + \cdots + p_{h_1} = 1 \text{ and } p_1 + \cdots + p_{i_1} + p_{h_1+1} + \cdots + p_m = 0 \text{ and } p_{i_1,1,j_1} = 0\}\big) \cup \big(\{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\}\big)$$

$$\equiv \big(\{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_1 + \cdots + p_{i_1} + p_{i_1+1} + \cdots + p_{h_1} + \cdots + p_m = 0 \text{ and } p_{i_1,1,j_1} = 0\}\big) \cup \big(\{\vec{t} \in \big(\textstyle\bigcup_{i=i_1+1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1+1,\ldots,h_1}}^{m} TP_i(S)\big) : p_{i_1,1,j_1} = 0\}\big) \cup \big(\{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\}\big)$$

$$\equiv \big(\{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_{i_1,1,j_1} = 0\}\big) \cup \big(\{\vec{t} \in \big(\bigcup_{i=i_1+1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1+1,\ldots,h_1}}^{m} TP_i(S)\big) : p_{i_1,1,j_1} = 0\}\big) \cup \big(\{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\}\big)$$

$$\equiv \{\vec{t} \in \big(\bigcup_{i=i_1+1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1+1,\ldots,h_1}}^{m} TP_i(S)\big) : p_{i_1,1,j_1} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 0\}$$

Now, the corresponding detection condition of Expression (70), denoted by $DC_{CORF \ltimes ENF}$, is "any point in $\big(\bigcup_{i=i_1+1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1+1,\ldots,h_1}}^{m} TP_i(S)\big)$ such that $p_{i_1,1,j_1} = 0$ or any point in $FP(S)$ such that $p_{i_1,j_1+1,k_{i_1}} = 0$". We use $TC_{CORF \ltimes ENF}$ to denote the set of all points that satisfy $DC_{CORF \ltimes ENF}$, that is $TC_{CORF \ltimes ENF} = \{\vec{t} \in \big(\bigcup_{i=i_1+1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i\neq i_1+1,\ldots,h_1}}^{m} TP_i(S)\big) : p_{i_1,1,j_1} = 0\} \cup \{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_{i_1}} = 0\}$.

(A) Relationship between DC_{CORF} and $DC_{CORF \ltimes ENF}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{CORF} can satisfy $DC_{CORF \ltimes ENF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{k}_{i_1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x^{i_1}_{k_{i_1}} = 0$. Hence $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1 + 0 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{k}_{i_1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \subseteq TC_{CORF}$. Note that, $\vec{t}$ satisfies $DC_{CORF \ltimes ENF}$ because $\vec{t}$ is a false point of S, $x^{i_1}_{k_{i_1}} = 0$ and $p_{i_1,j_1+1,k_{i_1}} = 0$ on $\vec{t}$.

(b) Some points in TC_{CORF} cannot satisfy $DC_{CORF \ltimes ENF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x^{i_1}_1 = 0$. Hence $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \subseteq TC_{CORF}$. Note that, $\vec{t}$ cannot satisfy $DC_{CORF \ltimes ENF}$ because $p_{i_1,j_1+1,k_{i_1}} = 1$ on $\vec{t}$.

Hence, only some, but not all, points satisfying DC_{CORF} can satisfy $DC_{CORF \ltimes ENF}$.

(B) Relationship between DC_{ENF} and $DC_{CORF \ltimes ENF}$:

For clarification and ease of discussion, let $X = \{\vec{t} \in \big(\bigcup_{i=i_1+1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1+1,\ldots,h_1}}^{m} TP_i(S)\big) :$ $p_{i_1,1,j_1} = 0\}$ and $Y = \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\}$.

We observe $TC_{ENF} \cap TC_{CORF \ltimes ENF}$

= $\big(X \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 0\}\big) \cap \big(X \cup \{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_{i_1}} = 0\}\big)$

= $X \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\}$ = $X \cup Y$.

The relationship between DC_{ENF} and $DC_{CORF \ltimes ENF}$ depends on whether both X and Y are empty. There are two cases:

(a) Both X and Y are empty. Then R6 holds. Here is an example. Let $S = ab + ac + \bar{a}\bar{b}$. If the first term ab is split into $a + b$ and then $b + ac$ is wrongly negated. $I_{CORF} = a + b + ac + \bar{a}\bar{b}$ and $I_{CORF \ltimes ENF} = a + \overline{b + ac} + \bar{a}\bar{b}$. Now, $TC_{ENF} = \{\vec{t} \in \big(TP_2(S) \setminus \bigcup_{\substack{i=1 \\ i\neq 2}}^{3} TP_i(S)\big) : a = 0\} \cup \{\vec{t} \in FP(S) : a = 0\} = \emptyset \cup \{010, 011\} = \{010, 011\}$; and $TC_{CORF \ltimes ENF} = \{\vec{t} \in \big(TP_2(S) \setminus \bigcup_{\substack{i=1 \\ i\neq 2}}^{3} TP_i(S)\big) : a = 0\} \cup \{\vec{t} \in FP(S) : b = 0\} = \emptyset \cup \{100\} = \{100\}$. Hence, all points satisfy TC_{ENF} do not satisfy $TC_{ENF \ltimes CORF}$.

(b) Any one of X and Y is not empty. Then R5 holds.

  (i) R5 holds when X is not empty because of the following reasons:

    (1) Some points in TC_{ENF} can satisfy $DC_{CORF \ltimes ENF}$. Since X is not empty, we can find a point $\vec{t}$ such that $\vec{t} \in X \subseteq TC_{ENF}$, $\vec{t}$ also satisfies $DC_{CORF \ltimes ENF}$.

    (2) Some points in TC_{ENF} cannot satisfy $DC_{CORF \ltimes ENF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x_1^{i_1} = 0$. Hence $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 0\} \subseteq TC_{ENF}$. Note that, $\vec{t}$ cannot satisfy $DC_{CORF \ltimes ENF}$ because $p_{i_1,j_1+1,k_{i_1}} = 1$ on $\vec{t}$.

  (ii) R5 holds when Y is not empty because of the following reasons

    (1) Some points in TC_{ENF} can satisfy $DC_{CORF \ltimes ENF}$. Since Y is not empty, we can find a point $\vec{t}$ such that $\vec{t} \in Y \subseteq TC_{ENF}$, $\vec{t}$ also satisfies $DC_{CORF \ltimes ENF}$.

(2) Some points in TC_{ENF} cannot satisfy $DC_{CORF \ltimes ENF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x_1^{i_1} = 0$. Hence $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 0\} \subseteq TC_{ENF}$. Note that, $\vec{t}$ cannot satisfy $DC_{CORF \ltimes ENF}$ because $p_{i_1,j_1+1,k_{i_1}} = 1$ on $\vec{t}$.

Hence, only some, but not all, points satisfying DC_{ENF} can satisfy $DC_{CORF \ltimes ENF}$.

5.2.3 CORF and TNF

In this section, we analyse the relationship between the double fault CORF⋉TNF and its corresponding single faults.

Let S be a Boolean expression in IDNF. The double fault expression given by Expression (73) can be derived from the situation where the i_1-th term, p_{i_1}, is wrongly implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$ and then the subexpression $p_{i_1,j_1+1,k_{i_1}}$ is wrongly negated.

As discussed in Section 2.2, if the i_1-th term, p_{i_1}, in S is wrongly implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$, the corresponding implementation, denoted by I_{CORF} will be equivalent to $p_1 + \cdots + p_{i_1-1} + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_{h_1} + \cdots + p_m$, and its detection condition, denoted by DC_{CORF}, is "any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1$". We use TC_{CORF} to denote the set of all points that satisfy DC_{CORF}, that is $TC_{CORF} = \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\}$.

Furthermore, given I_{CORF}, if the term $p_{i_1,j_1+1,k_{i_1}}$ is wrongly negated, the corresponding implementation, denoted by $I_{CORF \ltimes TNF}$, will be equivalent to $p_1 + \cdots + p_{i_1,1,j_1} + \overline{p_{i_1,j_1+1,k_{i_1}}} + p_{i_1+1} + \cdots + p_m$, and its detection condition with respect to I_{CORF}, denoted by DC_{TNF}, is "any point in $\{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m = 0\}$ or any point in $\left(\{\vec{t} \in \mathbb{B}^n : p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_m = 0\}\right)$". We use TC_{TNF} to denote the set of all points that satisfy DC_{TNF}, that is $TC_{TNF} = \left(\{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + \cdots + p_m = 0\}\right) \cup \left(\{\vec{t} \in\right.$

$\mathbb{B}^n : p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + \cdots + p_m = 0\}\big)$.

For ease of comparison, we express the condition in terms of S:

$$
\begin{aligned}
& \{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m = 0\} \cup \{\vec{t} \in \mathbb{B}^n : p_1 + \cdots + \\
& p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_m = 0\} \\
\equiv\ & \{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_m = 0\} \cup \{\vec{t} \in \mathbb{B}^n : \\
& p_1 + \cdots + p_{i_1,1,j_1} p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_m = 0 \text{ and } p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\} \\
\equiv\ & \{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_{i_1}} = 1\} \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\} \\
\equiv\ & \{\vec{t} \in FP(S) : p_{i_1} = 0 \text{ and } p_{i_1,j_1+1,k_{i_1}} = 1\} \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\} \\
& \text{Please note that if } \vec{t} \text{ is a false point of } S,\ p_{i_1} = 0. \\
\equiv\ & \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 0 \text{ and } p_{i_1,j_1+1,k_{i_1}} = 1\} \cup \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\} \\
\equiv\ & \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 0\}
\end{aligned}
$$

Now, the detection condition of Expression (73), denoted by $DC_{CORF \ltimes TNF}$, is "any point in $FP(S)$ such that $p_{i_1,j_1+1,k_{i_1}} = 0$". We use $TC_{CORF \ltimes TNF}$ to denote the set of all points that satisfy $DC_{CORF \ltimes TNF}$, that is $TC_{CORF \ltimes TNF} = \{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_{i_1}} = 0\}$.

(A) Relationship between DC_{CORF} and $DC_{CORF \ltimes TNF}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{CORF} can satisfy $DC_{CORF \ltimes TNF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{k}_{i_1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x^{i_1}_{k_{i_1}} = 0$. Hence, $p_{i_1,j_1+1,k_{i_1}} = 0$ and $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1 + 0 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{k}_{i_1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \subseteq TC_{CORF}$. Note that, $\vec{t}$ satisfies $DC_{CORF \ltimes TNF}$ because $\vec{t}$ is a false point of S and $p_{i_1,j_1+1,k_{i_1}} = 0$ on $\vec{t}$.

(b) Some points in TC_{CORF} cannot satisfy $DC_{CORF \ltimes TNF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x^{i_1}_1 = 0$. Hence, $p_{i_1,j_1+1,k_{i_1}} = 1$ and $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \subseteq TC_{CORF}$. Note that, $\vec{t}$

cannot satisfy $DC_{CORF \ltimes TNF}$ because $p_{i_1,j_1+1,k_{i_1}} = 1$ on $\vec{t}$.

Hence, only some, but not all, points satisfying DC_{CORF} can satisfy $DC_{CORF \ltimes TNF}$.

(B) Relationship between DC_{TNF} and $DC_{CORF \ltimes TNF}$:

For ease of references, let $X = \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\}$.

We observe $TC_{TNF} \cap TC_{CORF \ltimes TNF}$

= $\{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 0\} \cap \{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_{i_1}} = 0\}$

= $\{\vec{t} \in FP(S) : p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 0\}$

= X.

The relationship between DC_{TNF} and $DC_{CORF \ltimes TNF}$ depends on whether X is empty. There are two cases:

(a) X is empty. Then, R6 holds. Here is an example. Let $S = ab + \bar{a}\bar{b}$. If the first term ab is split into $a+b$ and b is then wrongly negated, $I_{CORF} = a + b + \bar{a}\bar{b}$ and $I_{CORF \ltimes TNF} = a + \bar{b} + \bar{a}\bar{b}$. Now, we have $TC_{TNF} = \{\vec{t} \in FP(S) : a = 0\}$ = $\{01\}$; and $TC_{CORF \ltimes TNF} = \{\vec{t} \in FP(S) : b = 0\}$ = $\{10\}$. Hence, all points satisfy TC_{ENF} do not satisfy $TC_{ENF \ltimes CORF}$.

(b) X is not empty. Then, R5 holds because of the following reasons:

(1) Some points in TC_{TNF} can satisfy $DC_{CORF \ltimes TNF}$. Since X is not empty, we can find a point $\vec{t}$ such that $\vec{t} \in X \subseteq TC_{TNF}$. Note that, $\vec{t}$ also satisfies $DC_{CORF \ltimes TNF}$.

(2) Some points in TC_{TNF} cannot satisfy $DC_{CORF \ltimes TNF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x_1^{i_1} = 0$. Hence, $p_{i_1,j_1+1,k_{i_1}} = 1$ and $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 0\} \subseteq TC_{TNF}$. Note that, $\vec{t}$ cannot satisfy $DC_{CORF \ltimes TNF}$ because $p_{i_1,j_1+1,k_{i_1}} = 1$ on $\vec{t}$.

5.2.4 CORF and TOF

In this section, we analyse the relationship between the double fault CORF⋈TOF and its corresponding single faults.

Let S be a Boolean expression in IDNF. The double fault expression given by Expression (76) can be derived from the situation where the i_1-th term, p_{i_1}, is wrongly implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$ and then the subexpression $p_{i_1,j_1+1,k_{i_1}}$ is wrongly omitted.

As discussed in Section 2.2, if the i_1-th term, p_{i_1}, in S is wrongly implemented as $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ where $p_{i_1} = p_{i_1,1,j_1} \cdot p_{i_1,j_1+1,k_{i_1}}$, the corresponding implementation, denoted by I_{CORF} will be equivalent to $p_1 + \cdots + p_{i_1-1} + p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_{h_1} + \cdots + p_m$, and its detection condition, denoted by DC_{CORF}, is "any point in $FP(S)$ such that $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}}$ = 1". We use TC_{CORF} to denote the set of all points that satisfy DC_{CORF}, that is $TC_{CORF} = \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\}$.

Furthermore, given I_{CORF}, if the subexpression $p_{i_1,j_1+1,k_{i_1}}$ is wrongly omitted, the corresponding implementation, denoted by $I_{CORF \bowtie TOF}$, will be equivalent to $p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m$, and its detection condition with respect to I_{CORF}, denoted by DC_{TOF}, is "any point in $\{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m = 0\}$. We use TC_{TOF} to denote the set of all points that satisfy DC_{TOF}, that is $TC_{TOF} = \{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m = 0\}$.

For ease of comparison, we express the condition in terms of S:

$$\begin{aligned}
& \{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} + p_{i_1+1} + \cdots + p_m = 0\} \\
\equiv\ & \{\vec{t} \in \mathbb{B}^n : p_{i_1,j_1+1,k_{i_1}} = 1 \text{ and } p_1 + \cdots + p_{i_1,1,j_1} p_{i_1,j_1+1,k_{i_1}} + p_{i_1+1} + \cdots + p_m = 0\} \\
\equiv\ & \{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_{i_1}} = 1\}.
\end{aligned}$$

Now, the detection condition of Expression (76), denoted by $DC_{CORF \bowtie TOF}$, is "any point in $FP(S)$ such that $p_{i_1,1,j_1} = 1$". We use $TC_{CORF \bowtie TOF}$ to denote the set of all points that satisfy $DC_{CORF \bowtie TOF}$, that is $TC_{CORF \ltimes TOF} = \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 1\}$.

(A) Relationship between DC_{CORF} and $DC_{CORF \ltimes TOF}$: The relationship R2 holds because of the following reasons:

(a) Some points in TC_{CORF} can satisfy $DC_{CORF \ltimes TOF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{k}_{i_1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x^{i_1}_{k_{i_1}} = 0$. Hence, $p_{i_1,1,j_1} = 1$ and $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1 + 0 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{k}_{i_1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \subseteq TC_{CORF}$. Note that, $\vec{t}$ satisfies $DC_{CORF \ltimes TOF}$ because $\vec{t}$ is a false point of S and $p_{i_1,1,j_1} = 1$ on $\vec{t}$.

(b) Some points in TC_{CORF} cannot satisfy $DC_{CORF \ltimes TOF}$. Since S is in IDNF, $NFP_{i,\bar{j}}(S) \neq \emptyset$ for all i and j. By definition of $NFP_{i_1,\bar{1}}(S)$, all literals in p_{i_1} evaluate to 1 except $x^{i_1}_1 = 0$. Hence, $p_{i_1,1,j_1} = 0$ and $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 0 + 1 = 1$ on $\vec{t}$. Therefore, we can find a point $\vec{t}$ such that $\vec{t} \in NFP_{i_1,\bar{1}}(S) \subseteq \{\vec{t} \in FP(S) : p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} = 1\} \subseteq TC_{CORF}$. Note that, $\vec{t}$ cannot satisfy $DC_{CORF \ltimes TOF}$ because $p_{i_1,1,j_1} = 0$ on $\vec{t}$.

Hence, only some, but not all, points satisfying DC_{CORF} can satisfy $DC_{CORF \ltimes TOF}$.

(B) Relationship between DC_{TOF} and $DC_{CORF \ltimes TOF}$: The relationship R6 holds because:

$$
\begin{aligned}
& TC_{TOF} \cap TC_{CORF \ltimes TOF} \\
= \; & \{\vec{t} \in FP(S) : p_{i_1,j_1+1,k_{i_1}} = 1\} \cap \{\vec{t} \in FP(S) : p_{i_1,1,j_1} = 1\} \\
= \; & \emptyset \text{ (because when } p_{i_1,1,j_1} = p_{i_1,j_1+1,k_{i_1}} = 1,\ p_{i_1} = p_{i_1,1,j_1} p_{i_1,j_1+1,k_{i_1}} = 1, \text{ hence, } S(\vec{t}) = 1 \\
& \text{which is impossible for any false point of } S\text{)}
\end{aligned}
$$

Hence, points satisfying DC_{TOF} cannot satisfy $DC_{CORF \ltimes TOF}$.

For ease of reading and understanding, Table 3 summarizes the relationship between the detection conditions of single fault classes and their corresponding double fault classes. Since our objective is to determine which double fault classes can always be detected by test cases that can detect individual single fault classes, we are particularly interested in R1 or R4. From Table 3, 15 out

Table 3: Relationship of detection conditions for single and double faults

(a) Double-fault expressions (1)–(27) due to double faults without ordering

		Fault class B				
		ENF	TNF	TOF	DORF	CORF
Fault class A	ENF	1: R2, R5; 2: R2, R6	3: R2, R5; 4: R2, R6	5: R1, R6; 6: R2, R6	7: R1, R6; 8: R2, R5; 9: R2, R6; 10: R2, R6	11: (R1,R2)[a], R4; 12: R2, R6
	TNF		13: R2, R5	14: R1, R6	15: R1, R6; 16: R2, R5	17: (R1,R2)[a], R4; 18: R2, R6
	TOF			19: R1, R4	20: R1, R4	21: (R1,R2,R3)[b], R4
	DORF				22: R1, R4; 23: R1, R4	24: (R1,R2,R3)[b], R4; 25: (R2,R3)[b], R5
	CORF					26: R1, R4; 27: R1, R4

(b) Four extra double-fault expressions (53), (70), (73) and (76) due to double faults with ordering

		Second fault class B			
		ENF	TNF	TOF	DORF
First fault class A	TOF				53: R1, R4
	CORF	70: R2, (R5, R6)[b]	73: R2, (R5, R6)[b]	76: R2, R6	

[a] "(R1,R2)" means that either R1 or R2 holds. Please see the corresponding discussion for details.
[b] The interpretation is similar to that of "(R1, R2)".

of 31 double-fault expressions have R1 or R4. These 15 expressions can always be detected by test cases that collectively detect the two individual single fault classes. For the remaining expressions, there is no guarantee that those test cases that detect each one of the two individual single faults will detect the double-fault expressions. It is interesting to note that R3 and R6 do not hold simultaneously. Hence, regarding the double fault classes studied in this report, there is always a chance that test cases that collectively detect individual single fault classes may also detect those double fault classes.

6 Comparing Existing Testing Strategies

Instead of concentrating on specified test set to detect studied double fault classes, we are more interested to find some test case selection strategies which can guarantee to detect them. Therefore, following questions are considered in this section:

> *Many test case selection strategies have been developed to detect all the individual single fault classes described in Section 2. Can these strategies also detect all the double fault classes considered in this report? If yes, which one can be used. If no, which double fault classes can be detected and which one cannot?*

Existing test case selection strategies for detecting faults in Boolean expressions include the BOR strategy [13, 14], the BASIC meaningful impact strategy (or simply the BASIC strategy) [17], the MUMCUT strategy [19] and the modified condition/decision coverage (MC/DC) criterion [3]. Since the BOR strategy requires every variable in the expression to occur only once, it is not widely applicable to all Boolean expressions in IDNF. It has been shown in [18] that MC/DC cannot guarantee to detect single-fault expressions due to TOF, DORF and CORF. As shown in [1, 19], both the BASIC and MUMCUT strategies can detect all single fault classes described in Section 2. Hence, we consider these two strategies further.

Given a Boolean expression S in IDNF, the BASIC strategy selects (1) a unique true point from every $UTP_i(S)$, and (2) a near false point from every $NFP_{i,\bar{j}}(S)$. The following three theorems prove that the test set selected by the BASIC strategy, and hence, any strategy that subsumes it, satisfies all detection conditions of all double-fault expressions considered in this report. A testing criterion $C1$ is said to *subsume* another criterion $C2$ if any test set that satisfies $C1$ must also satisfy $C2$. We need the following lemma to proceed.

Lemma 6.1 ([1, Theorem 1]) *Let S=$p_1 + \cdots + p_m$ be a Boolean specification in irredundant disjunctive normal form. Then, we have*

1. *$UTP_i(S) \neq \emptyset$ for all $i = 1, \ldots, m$*
2. *$NFP_{i,\bar{j}}(S) \neq \emptyset$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, k_i$, where k_i is the number of literals in the term p_i.*

Theorem 6.1 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Suppose that T is the set of near false points formed by selecting a near false point from $NFP_{i,\bar{j}}(S)$ for every i and j. Then T satisfies the following conditions:*

1. *There exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$.*
2. *For all possible i_2 and j_2 pair where $1 \leq i_2 \leq m$, $1 \leq j_2 < k_{i_2}$ and k_{i_2} is the number of literals of the i_2-th term p_{i_2} in S, there exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ and $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$.*
3. *For all possible i_1 and j_1 pair where $1 \leq i_1 \leq m$, $1 \leq j_1 < k_{i_1}$ and k_{i_1} denotes the number of literals of the i_1-th term p_{i_1} in S, there exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ and $p_{i_1,j_1+1,k_{i_1}} = 0$.*
4. *For all possible i_1 and j_1 pair where $1 \leq i_1 \leq m$, $1 \leq j_1 < k_{i_1}$ and k_{i_1} denotes the number of literals of the i_1-th term p_{i_1} in S, there exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ and $p_{i_1,1,j_1} = 1$.*

5. *For all possible i_1, i_2, j_1 and j_2 where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_1 < k_{i_1}$, $1 \leq j_2 < k_{i_2}$ and k_{i_1} and k_{i_2} denote the numbers of literals of the i_1-th and i_2-th terms, p_{i_1} and p_{i_2}, in S respectively, there exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ and $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = 1$.*

6. *For all possible i_1, j_1 and j_2 where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 < k_{i_1}$, and k_{i_1} denotes the number of literals of the i_1-th term p_{i_1} in S, there exists $\vec{t} \in T$ such that $\vec{t} \in FP(S)$ and $p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = 1$.*

Proof : By Lemma 6.1, T is non-empty because $NFP_{i,\bar{j}}(S)$ is non-empty for every possible i and j where $1 \leq i \leq m$, $1 \leq j \leq k_i$ and k_i is the number of literals of p_i in S.

1. By definition of T, any $\vec{t} \in T \subset NFP(S) \subset FP(S)$. Hence the result follows.

2. For any i_2 and j_2 pair where $1 \leq i_2 \leq m$, $1 \leq j_2 < k_{i_2}$ and k_{i_2} is the number of literals of p_{i_2}, there exists $\vec{t} \in T$ such that $\vec{t} \in NFP_{i_2,\bar{j_2}}(S) \subset FP(S)$ by definition of T and Lemma 6.1. Therefore, all literals of $p_{i_2}(x^{i_2}_1, \cdots, x^{i_2}_{k_{i_2}})$ evaluate to 1 except $x^{i_2}_{j_2}$ which evaluates to 0 on $\vec{t}$. Thus, $p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}} = x^{i_2}_1 \cdots x^{i_2}_{j_2} + x^{i_2}_{j_2+1} \cdots x^{i_2}_{k_{i_2}} = 0 + 1 = 1$. Hence, the result follows.

3. For any i_1 and j_1 pair where $1 \leq i_1 \leq m$, $1 \leq j_1 < k_{i_1}$ and k_{i_1} is the number of literals of p_{i_1} in S, there exists $\vec{t} \in T$ such that $\vec{t} \in NFP_{i_1,\overline{j_1+1}}(S) \subset FP(S)$ by definition of T and Lemma 6.1. Therefore, all literals of $p_{i_1}(x^{i_1}_1, \cdots, x^{i_1}_{k_{i_1}})$ evaluate to 1 except $x^{i_1}_{j_1+1}$ which evaluates to 0 on $\vec{t}$. Therefore, $p_{i_1,j_1+1,k_{i_1}} = x^{i_1}_{j_1+1} \cdots x^{i_1}_{k_{i_1}} = 0$. Hence, the result follows.

4. For any i_1 and j_1 pair where $1 \leq i_1 \leq m$, $1 \leq j_1 < k_{i_1}$ and k_{i_1} is the number of literals of p_{i_1} in S, there exists $\vec{t} \in T$ such that $\vec{t} \in NFP_{i_1,\overline{j_1+1}}(S) \subset FP(S)$ by definition of T and Lemma 6.1. Therefore, all literals of $p_{i_1}(x^{i_1}_1, \cdots, x^{i_1}_{k_{i_1}})$ evaluate to 1 except $x^{i_1}_{j_1+1}$ which evaluates to 0 on $\vec{t}$. Therefore, $p_{i_1,1,j_1} = x^{i_1}_1 \cdots x^{i_1}_{j_1} = 1$. Hence, the result follows.

5. For all possible i_1, i_2, j_1 and j_2 where $1 \leq i_1 < i_2 \leq m$, $1 \leq j_1 < k_{i_1}$, $1 \leq j_2 < k_{i_2}$ and k_{i_1} and k_{i_2} are the numbers of literals of p_{i_1} and p_{i_2} respectively, there exists $\vec{t} \in T$ such that $\vec{t} \in NFP_{i_1,\overline{j_1+1}}(S) \subset FP(S)$ by definition of T and Lemma 6.1. It should be noted that the

choice of $\vec{t}$ is independent of i_2 and j_2. Therefore, all literals of $p_{i_1}(x_1^{i_1}, \cdots, x_{k_{i_1}}^{i_1})$ evaluate to 1 except $x_{j_1+1}^{i_1}$ which evaluates to 0 on $\vec{t}$. Thus, $p_{i_1,1,j_1} + p_{i_1,j_1+1,k_{i_1}} + p_{i_2,1,j_2} + p_{i_2,j_2+1,k_{i_2}}$ = $x_1^{i_1} \cdots x_{j_1}^{i_1}$ + $x_{j_1+1}^{i_1} \cdots x_{k_{i_1}}^{i_1}$ + $x_1^{i_2} \cdots x_{j_2}^{i_2}$ + $x_{j_2+1}^{i_2} \cdots x_{k_{i_2}}^{i_2}$ = 1 + 0 + $x_1^{i_2} \cdots x_{j_2}^{i_2} + x_{j_2+1}^{i_2} \cdots x_{k_{i_2}}^{i_2}$ = 1. Hence, the result follows.

6. For all possible i_1, j_1 and j_2 where $1 \leq i_1 \leq m$, $1 \leq j_1 < j_2 < k_{i_1}$ and k_{i_1} is the number of literals of p_{i_1}, there exists $\vec{t} \in T$ such that $\vec{t} \in NFP_{i_1,\overline{j_1+1}}(S) \subset FP(S)$ by definition of T and Lemma 6.1. Therefore, all literals of $p_{i_1}(x_1^{i_1}, \cdots, x_{k_{i_1}}^{i_1})$ evaluate to 1 except $x_{j_1+1}^{i_1}$ which evaluates to 0 on $\vec{t}$. Therefore, $p_{i_1,1,j_1} + p_{i_1,j_1+1,j_2} + p_{i_1,j_2+1,k_{i_1}} = x_1^{i_1} \cdots x_{j_1}^{i_1} + x_{j_1+1}^{i_1} \cdots x_{j_2}^{i_1} + x_{j_2+1}^{i_1} \cdots x_{k_{i_1}}^{i_1} = 1 + 0 + 1 = 1$. Hence, the result follows.

□

Theorem 6.2 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Suppose that T is the set of unique true points formed by selecting a unique true point from $UTP_i(S)$ for every i. Then T satisfies the following conditions:*

1. *For any i_2 where $1 \leq i_2 \leq m$, there exists $\vec{t} \in T$ such that $\vec{t} \in UTP_{i_2}(S)$.*
2. *For any i_1, h_1, i_2 and h_2 where $1 \leq i_1 \leq i_2 < h_2 \leq h_1 \leq m$ and $\{i_2, i_2+1, \ldots, h_2-1, h_2\} \subsetneq \{i_1, i_1+1, \ldots, h_1\}$, there exists $\vec{t} \in T$ such that $\vec{t} \in \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$.*
3. *For any i_1 where $1 \leq i_1 < m$, there exists $\vec{t} \in T$ such that $\vec{t} \in TP_{i_1}(S) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1}}^{m} TP_i(S)\big)$.*
4. *For any i_1 and i_2 pair where $1 \leq i_1 < i_2 < m$, there exists $\vec{t} \in T$ such that $\vec{t} \in \big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S)\big)$.*
5. *For any i_1 and h_1 where $1 \leq i_1 < h_1 \leq m$, there exists $\vec{t} \in T$ such that $\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus$*

$\Big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S) \Big)$.

6. *For any i_1, i_2 and h_1 where $1 \le i_1 \le i_2 \le h_1 \le m$ and $i_1 < h_1$, there exists $\vec{t} \in T$ such that $\vec{t} \in \Big(\bigcup_{\substack{i=i_1 \\ i \neq i_2}}^{h_1} TP_i(S) \Big) \setminus \Big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S) \Big)$.*

7. *For any i_1 and h_1 where $1 \le i_1 < h_1 < m$, there exists $\vec{t} \in T$ such that $\vec{t} \in \Big(\bigcup_{i=i_1}^{h_1} TP_i(S) \Big) \setminus \Big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1+1}}^{m} TP_i(S) \Big)$.*

8. *For any i_1 and i_2 where $1 \le i_1 < i_2 < m$, there exists $\vec{t} \in T$ such that $\vec{t} \in \bigg(\Big(TP_{i_1}(S) \cup TP_{i_2}(S) \Big) \setminus \Big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2}}^{m} TP_i(S) \Big) \bigg) \cup \bigg(\Big(TP_{i_1}(S) \cup TP_{i_2+1}(S) \Big) \setminus \Big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_2+1}}^{m} TP_i(S) \Big) \bigg)$.*

9. *For any i_1 and i_2 where $1 \le i_1 < i_2 < m$, there exists $\vec{t} \in T$ such that $\vec{t} \in \Big(\bigcup_{\substack{i=i_1,i_1+1, \\ i_2,i_2+1}} TP_i(S) \Big) \setminus \bigg(\Big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S) \Big) \cup \Big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \Big) \cup \Big(TP_{i_2}(S) \cap TP_{i_2+1}(S) \Big) \bigg)$.*

10. *For any i_1 where $1 \le i_1 < m-1$, there exists $\vec{t} \in T$ such that $\vec{t} \in \Big(\bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S) \Big) \setminus \bigg(\Big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1,i_1+2}}^{m} TP_i(S) \Big) \cup \Big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S) \Big) \bigg)$.*

11. *For any i_1 where $1 < i_1 < m$, there exists $\vec{t} \in T$ such that $\vec{t} \in \bigg(\Big(TP_{i_1}(S) \cup TP_{i_1-1}(S) \Big) \setminus \Big(\bigcup_{\substack{i=1 \\ i \neq i_1-1,i_1}}^{m} TP_i(S) \Big) \bigg) \bigcup \bigg(\Big(TP_{i_1}(S) \cup TP_{i_1+1}(S) \Big) \setminus \Big(\bigcup_{\substack{i=1 \\ i \neq i_1,i_1+1}}^{m} TP_i(S) \Big) \bigg)$.*

Proof : By Lemma 6.1, T is non-empty because $UTP_i(S)$ is non-empty for every possible i where $1 \le i \le m$.

1. For any i_2 where $1 \le i_2 \le m$, there is a $\vec{t} \in T$ such that $\vec{t}$ in $UTP_{i_2}(S)$ by definition of T and Lemma 6.1.

2. Let i_1, h_1, i_2 and h_2 be such that $1 \leq i_1 \leq i_2 < h_2 \leq h_1 \leq m$ and $\{i_2, i_2+1, \ldots, h_2-1, h_2\} \subsetneqq \{i_1, i_1+1, \ldots, h_1\}$. Since the condition in the theorem is the detection condition of two subexpressions $p_{i_1} + \cdots + p_{h_1}$ and $p_{i_2} + \cdots + p_{h_2}$ being negated where $p_{i_2} + \cdots + p_{h_2}$ is contained in $p_{i_1} + \cdots + p_{h_1}$ and they are not identical, there is a term p_{i_3} in $p_{i_1} + \cdots + p_{h_1}$ but not in $p_{i_2} + \cdots + p_{h_2}$. By definition of T and Lemma 6.1, there is a $\vec{t} \in T$ such that $\vec{t} \in UTP_{i_3}(S)$. We then have $\vec{t} \in UTP_{i_3}(S) \subset TP_{i_3}(S) \subset \big(\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)\big)$ because $i_3 \in \{i_1, \cdots, h_1\} \setminus \{i_2, \cdots, h_2\}$. Since $\vec{t} \in UTP_{i_3}(S)$, $\vec{t} \notin TP_i(S)$ for any $i \neq i_3$. Hence, $\vec{t} \in \Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_3}}^{m} TP_i(S)\big) \Big) \subset \Big(\big(\bigcup_{\substack{i=i_1 \\ i \neq i_2,\ldots,h_2}}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big) \Big)$. The result follows.

3. For any i_1 where $1 \leq i_1 < m$, there is a $\vec{t} \in T$ such that $\vec{t} \in UTP_{i_1}(S)$ by definition of T and Lemma 6.1. Therefore, $\vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S)$. Since $\vec{t} \in UTP_{i_1}(S)$, $\vec{t} \notin TP_i(S)$ for any $i \neq i_1$. Hence, $\vec{t} \in \big(TP_{i_1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big) \subset \big(TP_{i_1}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_1+1}}^{m} TP_i(S)\big)$. The result follows.

4. For any i_1 and i_2 pair where $1 \leq i_1 < i_2 < m$, there is a $\vec{t} \in T$ such that $\vec{t} \in UTP_{i_1}(S)$ by definition of T and Lemma 6.1. It should be noted that the choice of $\vec{t}$ is independent of i_2. Therefore, $\vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset TP_{i_1}(S) \cup TP_{i_2}(S)$. Since $\vec{t} \in UTP_{i_1}(S)$, $\vec{t} \notin TP_i(S)$ for any $i \neq i_1$. Hence, $\vec{t} \in \big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big) \subset \big(TP_{i_1}(S) \cup TP_{i_2}(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1, i_2}}^{m} TP_i(S)\big)$. The result follows.

5. For any i_1 and h_1 where $1 \leq i_1 < h_1 \leq m$, there is a $\vec{t} \in T$ such that $\vec{t} \in UTP_{i_1}(S)$ by definition of T and Lemma 6.1. It should be noted that the choice of $\vec{t}$ is independent of h_1. Therefore, $\vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big)$. Since $\vec{t} \in UTP_{i_1}(S)$, $\vec{t} \notin TP_i(S)$ for any $i \neq i_1$. Hence, $\vec{t} \in \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1}}^{m} TP_i(S)\big) \subset \big(\bigcup_{i=i_1}^{h_1} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1 \\ i \neq i_1,\ldots,h_1}}^{m} TP_i(S)\big)$. The result follows.

6. Let i_1, i_2 and h_1 be such that $1 \leq i_1 \leq i_2 \leq h_1 \leq m$ and $i_1 < h_1$. Since the condition in the theorem is the detection condition of both the subexpression $p_{i_1} + \cdots + p_{h_1}$ and the term p_{i_2}

being negated where p_{i_2} is contained in $p_{i_1}+\cdots+p_{h_1}$ and $i_1<h_1$, there is a term p_{i_3} in $p_{i_1}+\cdots+p_{h_1}$ that is different from p_{i_2}. By definition of T and Lemma 6.1, there is a $\vec{t}\in T$ such that $\vec{t}\in UTP_{i_3}(S)$. Therefore, $\vec{t}\in UTP_{i_3}(S)\subset TP_{i_3}(S)\subset\big(\bigcup_{\substack{i=i_1\\i\neq i_2}}^{h_1}TP_i(S)\big)$. Furthermore, since $\vec{t}\in UTP_{i_3}(S)$, $\vec{t}\notin TP_i(S)$ for any $i\neq i_3$. Hence, $\vec{t}\in\big(\bigcup_{\substack{i=i_1\\i\neq i_2}}^{h_1}TP_i(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_3}}^{m}TP_i(S)\big)\subset\big(\bigcup_{\substack{i=i_1\\i\neq i_2}}^{h_1}TP_i(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_1,\ldots,h_1}}^{m}TP_i(S)\big)$. The result follows.

7. For any i_1 and h_1 where $1\leq i_1<h_1<m$, there is a $\vec{t}\in T$ such that $\vec{t}\in UTP_{i_1}(S)$ by definition of T and Lemma 6.1. It should be noted that the choice of $\vec{t}$ is independent of h_1. Therefore, $\vec{t}\in UTP_{i_1}(S)\subset TP_{i_1}(S)\subset\big(\bigcup_{i=i_1}^{h_1}TP_i(S)\big)$. Since $\vec{t}\in UTP_{i_1}(S), \vec{t}\notin TP_i(S)$ for any $i\neq i_1$. Hence, $\vec{t}\in\big(\bigcup_{i=i_1}^{h_1}TP_i(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_1}}^{m}TP_i(S)\big)\subset\big(\bigcup_{i=i_1}^{h_1}TP_i(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_1,\ldots,h_1+1}}^{m}TP_i(S)\big)$. The result follows.

8. For any i_1 and i_2 where $1\leq i_1<i_2<m$, there is a $\vec{t}\in T$ such that $\vec{t}\in UTP_{i_1}(S)$ by definition of T and Lemma 6.1. It should be noted that the choice of $\vec{t}$ is independent of i_2. Thus, $\vec{t}\in UTP_{i_1}(S)\subset TP_{i_1}(S)\cup TP_{i_2}(S)$. Since $\vec{t}\in UTP_{i_1}(S)$, $\vec{t}\notin TP_i(S)$ for $i\neq i_1$. Hence, $\vec{t}\in\Big(\big(TP_{i_1}(S)\cup TP_{i_2}(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_1}}^{m}TP_i(S)\big)\Big)\subset\Big(\big(TP_{i_1}(S)\cup TP_{i_2}(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_1,i_2}}^{m}TP_i(S)\big)\Big)\subset\Big(\big(TP_{i_1}(S)\cup TP_{i_2}(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_1,i_2}}^{m}TP_i(S)\big)\Big)\bigcup\Big(\big(TP_{i_1}(S)\cup TP_{i_2+1}(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_1,i_2+1}}^{m}TP_i(S)\big)\Big)$.

9. For any i_1 and i_2 where $1\leq i_1<i_2<m$, there is a $\vec{t}\in T$ such that $\vec{t}\in UTP_{i_1}(S)$ by definition of T and Lemma 6.1. It should be noted that the choice of $\vec{t}$ is independent of i_2. Thus, $\vec{t}\in UTP_{i_1}(S)\subset TP_{i_1}(S)\subset\big(\bigcup_{\substack{i=i_1,i_1+1\\i_2,i_2+1}}TP_i(S)\big)$. Since $\vec{t}\in UTP_{i_1}(S)$, $\vec{t}\notin TP_i(S)$ for any $i\neq i_1$. Hence, $\vec{t}\in\Big(\big(\bigcup_{\substack{i=i_1,i_1+1\\i_2,i_2+1}}TP_i(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_1}}^{m}TP_i(S)\big)\Big)\subset\Big(\big(\bigcup_{\substack{i=i_1,i_1+1\\i_2,i_2+1}}TP_i(S)\big)\setminus\big(\bigcup_{\substack{i=1\\i\neq i_1,i_1+1\\i_2,i_2+1}}^{m}TP_i(S)\big)\Big)$. Since $\vec{t}\in UTP_{i_1}(S)$, $\vec{t}$ does not belong to any of these sets $TP_{i_1+1}(S)$, $TP_{i_2}(S)$ and $TP_{i_2+1}(S)$.

Therefore, $\vec{t} \notin \Big(\big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_2}(S) \cap TP_{i_2+1}(S)\big)\Big)$. Hence, $\vec{t} \in \bigcup_{\substack{i=i_1,i_1+1,\\ i_2,i_2+1}} TP_i(S) \setminus \Big(\big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big)\Big)$. Therefore, we have $\vec{t} \in \bigcup_{\substack{i=i_1,i_1+1,\\ i_2,i_2+1}} TP_i(S) \setminus \Big(\big(\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_2,i_2+1}}^{m} TP_i(S)\big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S)\big)\Big)$. The result follows.

10. For any i_1 where $1 \leq i_1 < m-1$, there is a $\vec{t} \in T$ such that $\vec{t} \in UTP_{i_1}(S)$ by definition of T and Lemma 6.1. Therefore, $\vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset \big(\bigcup_{i=i_1,i_1+1,i_i+2} TP_i(S)\big)$. Since $\vec{t} \in UTP_{i_1}(S)$, $\vec{t} \notin TP_i(S)$ for any $i \neq i_1$. Hence, $\vec{t} \in \big(\bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1}}^{m} TP_i(S)\big) \subset \big(\bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_1+2}}^{m} TP_i(S)\big)$. Since $\vec{t} \in UTP_{i_1}(S)$, $\vec{t}$ does not belong to any of these two sets, $TP_{i_1+1}(S)$ and $TP_{i_1+2}(S)$. Thus, $\vec{t} \notin \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)$. Hence, $\vec{t} \in \big(\bigcup_{i=i_1,i_1+1,i_2,i_2+1} TP_i(S)\big) \setminus \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)$. Thus, we have $\vec{t} \in \big(\bigcup_{i=i_1,i_1+1,i_1+2} TP_i(S)\big) \setminus \Big(\big(\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1,i_1+2}}^{m} TP_i(S)\big) \cup \big(TP_{i_1}(S) \cap TP_{i_1+1}(S) \cap TP_{i_1+2}(S)\big)\Big)$. The result follows.

11. For any i_1 where $1 < i_1 < m$, there is a $\vec{t} \in T$ such that $\vec{t} \in UTP_{i_1}(S)$ by definition of T and Lemma 6.1. Thus, $\vec{t} \in UTP_{i_1}(S) \subset TP_{i_1}(S) \subset \big(TP_{i_1}(S) \cup TP_{i_1-1}(S)\big)$. Since $\vec{t} \in UTP_{i_1}(S)$, $\vec{t} \notin TP_i(S)$ for $i \neq i_1$. Hence, $\vec{t} \in \Big(\big(TP_{i_1}(S) \cup TP_{i_1-1}(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1}}^{m} TP_i(S)\big)\Big) \subset \Big(\big(TP_{i_1}(S) \cup TP_{i_1-1}(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1-1,i_1}}^{m} TP_i(S)\big)\Big) \subset \Big(\big(TP_{i_1}(S) \cup TP_{i_1-1}(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1-1,i_1}}^{m} TP_i(S)\big)\Big) \bigcup \Big(\big(TP_{i_1}(S) \cup TP_{i_1+1}(S)\big) \setminus \big(\bigcup_{\substack{i=1\\ i\neq i_1,i_1+1}}^{m} TP_i(S)\big)\Big)$. The result follows.

□

Table 4: Conclusions in Theorems 6.1 and 6.2 satisfying detection conditions in Table 2

Double-fault expression	Detection condition	Conclusion in	
		Theorem 6.1	Theorem 6.2
(1), (3), (5)–(7), (9)–(11), (13)–(15), (17)	(C2)	(1)	–
(2)	(C1)	–	(2)
(4)	(C1)	–	(6)
(8)	(C1)	–	(7)
(12)	(C1)	–	(5)
(16)	(C1)	–	(3)
(18)	(C1)	–	(1)
(19)	(C1)	–	(4)
(20)	(C1)	–	(8)
(21)	(C2)	(2)	–
(22)	(C1)	–	(9)
(23)	(C1)	–	(10)
(24)	(C2)	(2)	–
(25)	(C2)	(4)	–
(26)	(C1)	(5)	–
(27)	(C1)	(6)	–
(53)	(C1)	–	(11)
(70)	(C2)	(3)	–
(73)	(C1)	(3)	–
(76)	(C2)	(4)	–

Theorem 6.3 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Then, the BASIC meaningful impact strategy can detect all 31 double-fault expressions considered in this report.*

Proof : As a reminder, the BASIC meaningful impact strategy selects (1) one unique true point from every $UTP_i(S)$, and (2) one near false point from every $NFP_{i,j}(S)$. Table 4 indicates that all detection conditions of all 31 double-fault expressions in Table 2 can be collectively satisfied by the corresponding conclusions in Theorems 6.1 and 6.2. Hence, the result follows. □

Theorem 6.4 *Let $S = p_1 + \cdots + p_m$ be a Boolean expression in irredundant disjunctive normal form. Then, any test case selection strategy that subsumes the BASIC meaningful impact strategy can detect all double-fault expressions considered in this report.*

Proof : By Theorem 6.3, the BASIC meaningful impact strategy can detect all 31 double-fault expressions considered in this report. As a result, any test case selection strategy that subsumes the BASIC meaningful impact strategy can detect them. □

Among the existing test case selection strategies proposed in research literature, those that subsume the BASIC strategy include the MANY-A strategy, the MAX-A strategy and the MUMCUT strategy [17, 19]. By Theorem 6.4, all these strategies can also detect all double fault classes considered in this report. The empirical study in [17] shows that, for a group of Boolean expressions under study, the BASIC, MANY-A and MAX-A strategies require, on average, test sets of sizes 9.8%, 19.3% and 40.6%, respectively, of the entire input domain. As reported in [19], the sizes of test sets generated by the MUMCUT strategy for the same group of Boolean expressions are, on average, approximately 12.0% of the entire input domain. Hence, both the BASIC and MUMCUT strategies generate much smaller test sets than the MANY-A and MAX-A strategies do, and that they can detect all double faults related to terms in Boolean expressions. Although the MUMCUT strategy requires slightly larger test sets, it can detect more single fault classes (such as literal insertion fault and literal reference fault) than that of the BASIC strategy [1, 19].

7 Conclusion

In this report, we study the detection conditions on double faults related to terms within a Boolean expression. For double fault related to terms, 49 out of 53 faulty expressions of double faults with ordering are equivalent to the 27 distinct double-fault expressions due to double faults without ordering, and the 4 remaining double-fault expressions are not equivalent to any of 27 faulty expressions. Altogether, there are 31 different double-fault expressions among all double fault classes considered in this report.

We also study the fault coupling between single fault classes and their corresponding double fault classes via analysing the relationship between detection conditions of single and double fault classes. We find that 15 out of the 31 double-fault expressions can always be detected by test cases that detect single fault classes. Moreover, for the remaining 16 double-fault expressions, some but not all test cases that detect each individual single fault class will also detect the double-fault expressions.

Based on the detection conditions, we prove that any test case selection strategy that subsumes the BASIC meaningful impact strategy can detect all double fault classes considered in this report. This is very interesting because none of these strategies were originally developed for detecting double faults. Among existing test case selection strategies based on Boolean expressions, both the BASIC strategy and the MUMCUT strategy generate much smaller test sets than other strategies that subsume the BASIC strategy.

Some other classes of single faults related to literals in Boolean expressions have not been studied in this report. We are extending our work to explore detection conditions of double faults related to literals and to further analyse whether existing test case selection strategies are able to detect all these double fault classes. We intend to complete the analysis of detection conditions so as to understand more precisely the behaviour of multiple faults for Boolean expressions.

References

[1] T. Y. Chen and M. F. Lau. Test case selection strategies based on boolean specifications. *Software Testing, Verification and Reliability*, 11(3):165 – 180, 2001.

[2] T. Y. Chen, T. H. Tse, and Z. Q. Zhou. Fault-based testing without the need of oracles. *Information and Software Technology*, 45(1):1 – 9, 2003.

[3] J. J. Chilenski and S. P. Miller. Applicability of modified condition/decision coverage to software testing. *Software Engineering Journal*, 9(5):193 – 200, 1994.

[4] R. DeMillo, R. Lipton, and F. Sayward. Hints on test data selection: Help for the practicing programmer. *IEEE Computer*, 11(4):34–41, 1978.

[5] K. S. How Tai Wah. A theoretical study of fault coupling. *Software Testing, Verification and Reliability*, 10(1):3–45, 2000.

[6] D. Kuhn. Fault classes and error detection capability of specification-based testing. *ACM Transactions on Software Engineering and Methodology*, 8(4):411–424, 1999.

[7] M. F. Lau and Y. Liu. Classification and relationship of double faults related to terms in Boolean expressions. Technical Report SUTICT-TR2006.01, Swinburne University of Technology, 2006.

[8] M. F. Lau and Y. T. Yu. On the relationship between single and double faults in logical expression. In *Proceedings of the 15th International Symposium on Software Reliability Engineering (ISSRE 2004)*, pages 41 – 42, 2004.

[9] M. F. Lau and Y. T. Yu. An extended fault class hierarchy for specification-based testing. *ACM Transactions on Software Engineering and Methodology*, 14(3):247 – 276, 2005.

[10] B. Marick. Two experiments in software testing. Technical Report Technical Report UIUCDCS-R-90-1644, University of Illinois at Urbana-Champaign, 1990.

[11] L. J. Morell. A theory of fault-based testing. *IEEE Transactions on Software Engineering*, 16(8):844–857, 1990.

[12] A. J. Offutt. Investigations of the software testing coupling effect. *ACM Transactions on Software Engineering and Methodology*, 1(1):5–20, 1992.

[13] K. C. Tai. Condition-based software testing strategies. In *Proceedings of COMPSAC'90*, pages 564–569, 1990.

[14] K. C. Tai and H. K. Su. Test generation for Boolean expressions. In *Proceedings of COMPSAC'87*, pages 278–284, 1987.

[15] T. Tsuchiya and T. Kikuno. On fault classes and error detection capability of specification-based testing. *ACM Transactions on Software Engineering and Methodology*, 11(1):58 – 62, 2002.

[16] D. Wallace and D. Kuhn. Failure modes in medical device software: an analysis of 15 years of recall data. *International Journal of Reliability, Quality, and Safety Engineering*, 8(4), 2001.

[17] E. Weyuker, T. Goradia, and A. Singh. Automatically generating test data from a Boolean specification. *IEEE Transactions on Software Engineering*, 20(5):353–363, 1994.

[18] Y. T. Yu and M. F. Lau. A comparison of MC/DC, MUMCUT and several other coverage criteria for logical decisions. *Journal of Systems and Software*, 79(5):577–590, 2006.

[19] Y. T. Yu, M. F. Lau, and T. Y. Chen. Automatic generation of test cases from Boolean specifications using the MUMCUT strategy. *Journal of Systems and Software*, 79(6):820–840, 2006.