Appendix 1. Analytic solutions for the independent Lyapunov exponents of a dilute system under planar elongational flow.

We present a simple analytical proof of the fact that, in the low density regime, the independent Lyapunov exponents can be approximated by $\dot{\varepsilon}$, $-\dot{\varepsilon}$, $\dot{\varepsilon}-\langle\alpha\rangle$ and $-\dot{\varepsilon}-\langle\alpha\rangle$ for system under planar elongational flow. Consider Eqs. (1.35) in the dilute regime $(\mathbf{F}_i \simeq \mathbf{0})$:

$$\begin{aligned}\dot{\mathbf{r}}_i &= \frac{\mathbf{p}_i}{m_i} + \dot{\varepsilon}\left(\mathbf{i}x_i + \mathbf{j}y_i\right) \\ \dot{\mathbf{p}}_i &= -\dot{\varepsilon}\left(\mathbf{i}p_{xi} + \mathbf{j}p_{yi}\right) - \alpha\mathbf{p}_i\end{aligned} \tag{A1.1}$$

and assume that the thermostat multiplier α is constant. The equations of motion for the particles are then independent and each particle can be considered separately. From now on, we impose $m_i = m = 1$ for every i and then drop the subscript i that refers to the particle.

The equations of motion for the components of the tangent vector $\delta\Gamma$ (see Section 3.2) become:

$$\delta\dot{x} = \delta p_x + \dot{\varepsilon}\delta x \tag{A1.2}$$

$$\delta\dot{y} = \delta\dot{p}_y - \dot{\varepsilon}\delta y \tag{A1.3}$$

$$\delta\dot{p}_x = -(\dot{\varepsilon}+\alpha)\delta p_x \tag{A1.4}$$

$$\delta\dot{p}_y = (\dot{\varepsilon}-\alpha)\delta p_y \tag{A1.5}$$

Equations (A1.3) and (A1.4) can be solved to give:

$$\delta p_x(t) = \exp[-(\dot{\varepsilon}+\alpha)t]\delta p_x(0) \tag{A1.6}$$

$$\delta p_y(t) = \exp[(\dot{\varepsilon}-\alpha)t]\delta p_y(0) \tag{A1.7}$$

where $\delta p_x(0)$ and $\delta p_y(0)$ are the initial displacement at time t = 0 in the x and y directions of the momentum, respectively. These can then be substituted into (A1.2) and (A1.3) to give:

$$\delta\dot{x}(t) = \exp[-(\dot{\varepsilon}+\alpha)t]\delta p_x(0) + \dot{\varepsilon}\delta x(t) \qquad \text{(A1.8)}$$

$$\delta\dot{y}(t) = \exp[(\dot{\varepsilon}-\alpha)t]\delta p_y(0) - \dot{\varepsilon}\delta y(t) \qquad \text{(A1.9)}$$

and (A1.8) and (A1.9) can be solved to give

$$\delta x(t) = -\frac{\delta p_x(0)}{2\dot{\varepsilon}+\alpha}\left(\exp\left(-(\dot{\varepsilon}+\alpha)t\right) - \exp\left(\dot{\varepsilon}t\right)\right) + \delta x(0)\exp\left(\dot{\varepsilon}t\right) \qquad \text{(A1.10)}$$

$$\delta y(t) = \frac{\delta p_y(0)}{2\dot{\varepsilon}+\alpha}\left(\exp\left((\dot{\varepsilon}-\alpha)t\right) - \exp\left(-\dot{\varepsilon}t\right)\right) + \delta y(0)\exp\left(-\dot{\varepsilon}t\right) \qquad \text{(A1.11)}$$

A suitable choice of an orthonormal basis of initial vectors (Benettin *et al.*, 1976; Shimada and Nagashima, 1979; Benettin *et al.*, 1980a, 1980b) is represented by those with independent displacements along the x, y, p_x and p_y directions, so that, omitting the normalization parameters, we can write

$$\begin{aligned}
\delta\boldsymbol{\Gamma}_x(0) &= \left(\delta x(0), 0, 0, 0\right) \\
\delta\boldsymbol{\Gamma}_y(0) &= \left(0, \delta y(0), 0, 0\right) \\
\delta\boldsymbol{\Gamma}_{p_x}(0) &= \left(0, 0, \delta p_x(0), 0\right) \\
\delta\boldsymbol{\Gamma}_{p_y}(0) &= \left(0, 0, 0, \delta p_y(0)\right)
\end{aligned}$$

Consider the first vector, when the initial displacement in the x direction is $\delta x(0) = a$, where a is small (in numerical simulations and in reduced units $a = 0.0001$, according to (Morriss, 1988)) and all other initial displacements along y, p_x and p_y are zero. $\delta x(0)$ will grow exponentially in time according to (A1.10), with a rate $\dot{\varepsilon}$. Clearly, all other components will remain zero according to (A1.6), (A1.7) and (A1.11), so that the exponent associated with $\delta\boldsymbol{\Gamma}_x$ will have a value $\dot{\varepsilon}$.

A similar argument can be used for $\delta\boldsymbol{\Gamma}_y$: if $\delta y(0) = a$, it will grow exponentially in time with a rate $-\dot{\varepsilon}$. Again, all other initial displacements along x, p_x and p_y are zero, so they will remain zero. The associated Lyapunov exponent will therefore have a value $-\dot{\varepsilon}$.

The case for $\delta\boldsymbol{\Gamma}_{p_x}$ is different: if $\delta p_x(0) = a$, this component will grow exponentially with a rate $-(\dot{\varepsilon}+\alpha)$. The components in the y and p_y directions remain zero, but the x component will grow exponentially with a rate $\dot{\varepsilon}$, according to the dominant exponent in the first term of (A1.10). But, the only non-zero component of the resultant vector, which must remain orthogonal to $\delta\boldsymbol{\Gamma}_x$, will grow exponentially with the rate $-(\dot{\varepsilon}+\alpha)$, and therefore there is a Lyapunov exponent $-(\dot{\varepsilon}+\alpha)$ associated with $\delta\boldsymbol{\Gamma}_{p_x}$.

For $\delta\boldsymbol{\Gamma}_{p_y}$, if $\delta p_x(0) = a$, this component will grow exponentially with a rate $\dot{\varepsilon}-\alpha$. The components in the x and p_x directions remain zero, but δy will grow exponentially with a rate either $-\dot{\varepsilon}$ or $\dot{\varepsilon}-\alpha$ because of (A1.11), depending on which one is larger. As above, the only non-zero component of the resultant vector, which again has to be orthogonal to $\delta\boldsymbol{\Gamma}_y$, will grow exponentially with the rate $\dot{\varepsilon}-\alpha$ and there is a corresponding Lyapunov exponent with this value.

The values $\dot{\varepsilon}$, $-\dot{\varepsilon}$, $\dot{\varepsilon}-\langle\alpha\rangle$ and $-\dot{\varepsilon}-\langle\alpha\rangle$ for the independent exponents in the dilute regime are obtained from the above arguments, approximating α with its time average $\langle\alpha\rangle$.