HMS111
Engineering Mathematics 1

Functions and Graphs

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What are we going to learn ?

Find the composition of two functions

Form equations and graph straight lines

Form equations and graph circular functions

Form equations and graph elliptical functions

What are we going to learn ?

Form equations and graph hyperbolic functions

Form equations and graph parabolic functions

Find partial fractions from a given function

Find inverse functions

What are we going to learn ?

Do calculations with exponential and log. functions

Do calculations involving hyperbolic functions

Do calculations involving trig. and inverse trig. functions

Functions

"f(x)" is powerful !!

A shift from "rule of thumb", ad hoc calculations and limitations of Greek geometry to generalized methods of analysis

Thank you Rene Descartes !!!

Rene Descartes (1596-1650)

René Descartes. Portrait after Frans Hals, 1648

Mathematician, philosopher and soldier (Spy ??)

Father of modern Philosophy (Rationalist)

Founder of co-ordinate geometry & modern algebraic notation

“I think, therefore I am” Rene Descartes

“Rene Descarte was a drunken fart, I drink,therefore I am”

Monty Python

What do we do with functions ?

Find solutions analytically where possible

Find solutions using numerical techniques

Use graphs to visualise and understand the functions we have formed

Form new functions and relationships from existing functions (e.g. calculus)

Variable quantities

$y = f(x)$ = dependent variable

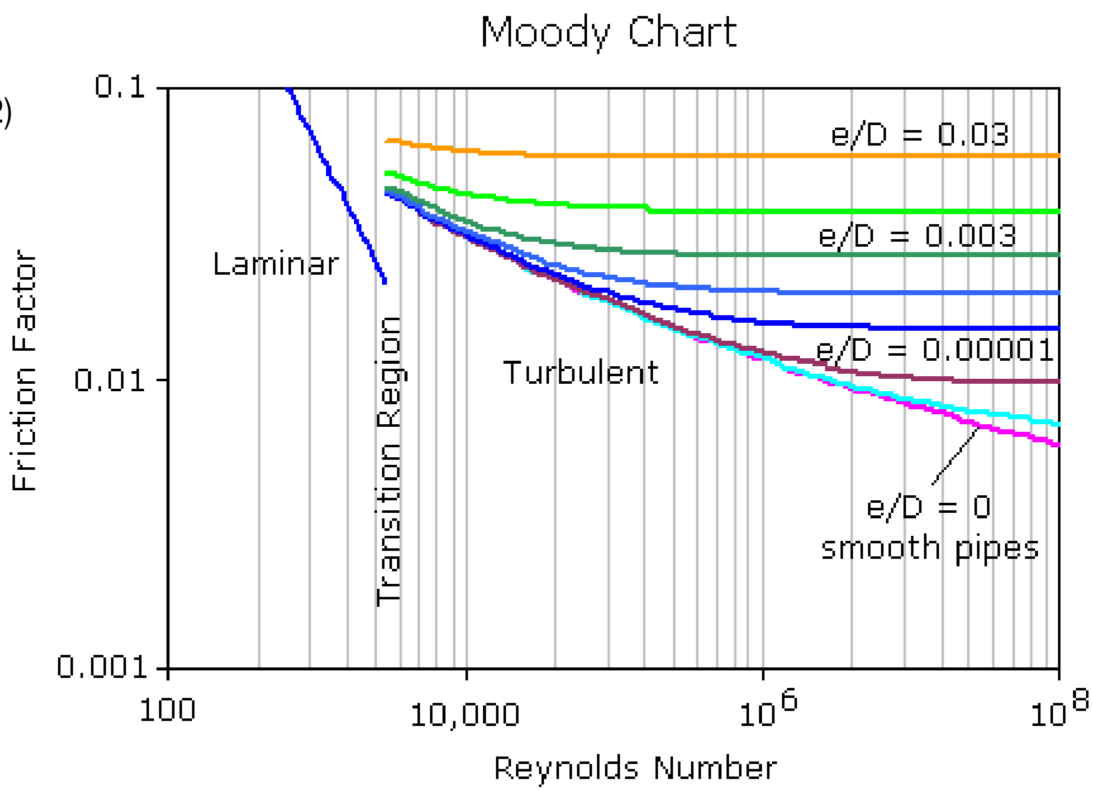
x = independent variable

$f(x)$ is often defined for a specific range

Moody Chart

N_{Re} = DV❒/❍   F = 2👎PD/(L ❒V²)

Where
- D = diameter of pipe
- V = velocity
- 👎P = pressure drop
- ❒ = density of liquid
- ❍ = viscosity of liquid
- L = length of the pipe



Defining domains for functions

Defining domains for the function based on the:

(i) the applicability of that function, and

(ii) the function making “sense”

e.g. avoiding dividing by infinity and “real number” solutions

Domains

Consider $f(x) = 1/(2-x)$

$f(x) = (x^2 - 4)^{1/2}$

$f(x) = 1/(x^2 - 9)^{1/2}$

Composition of functions

Combining functions is common:

$$f(x) = x^2$$

$$g(x) = 6x + 4$$

$$f(g(x)) = (6x + 4)^2$$

$$g(f(x)) = 6x^2 + 4$$

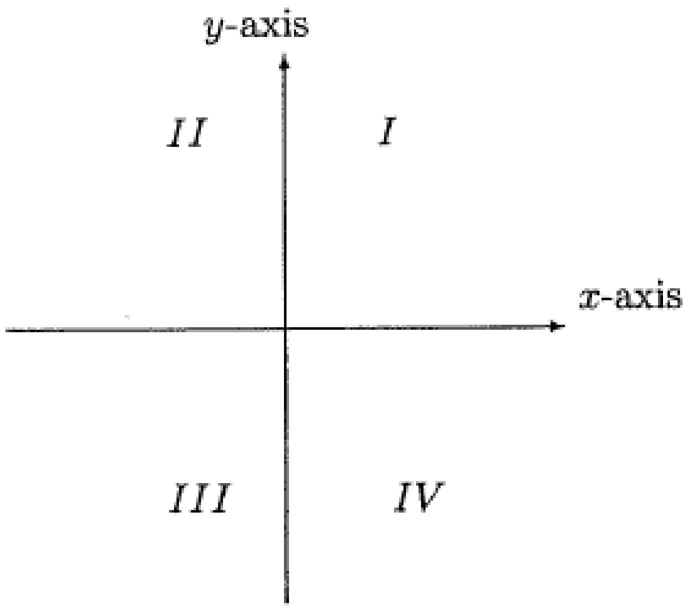
Graphs and curves

Straight lines

$y = mx + c$ $\qquad$ $m = \text{rise/run} = (y_2 - y_1)/(x_2 - x_1)$

$c = y$ intercept

Graphs and curves



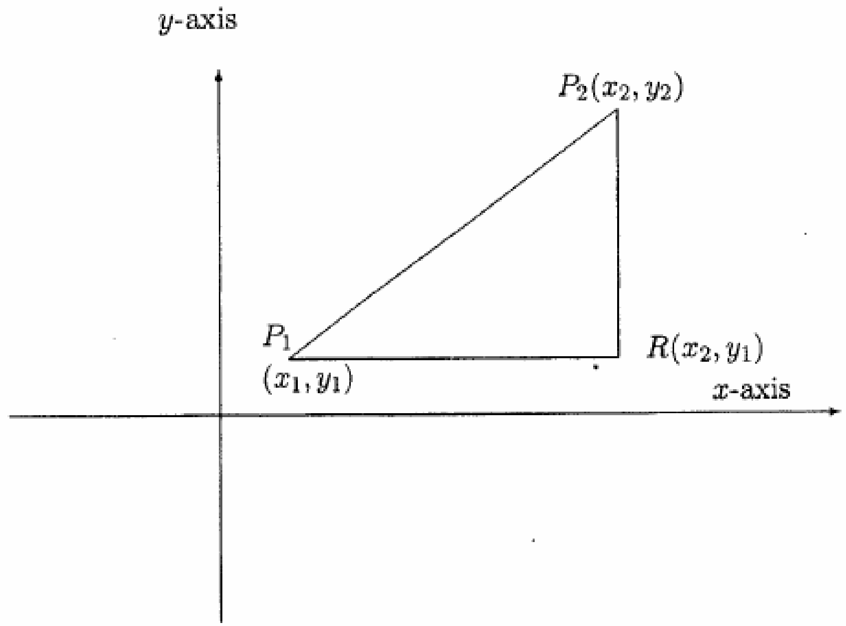


Image from Barling et al. “Engineering Mathematics 1 Student Notes”, 2009

Family of curves

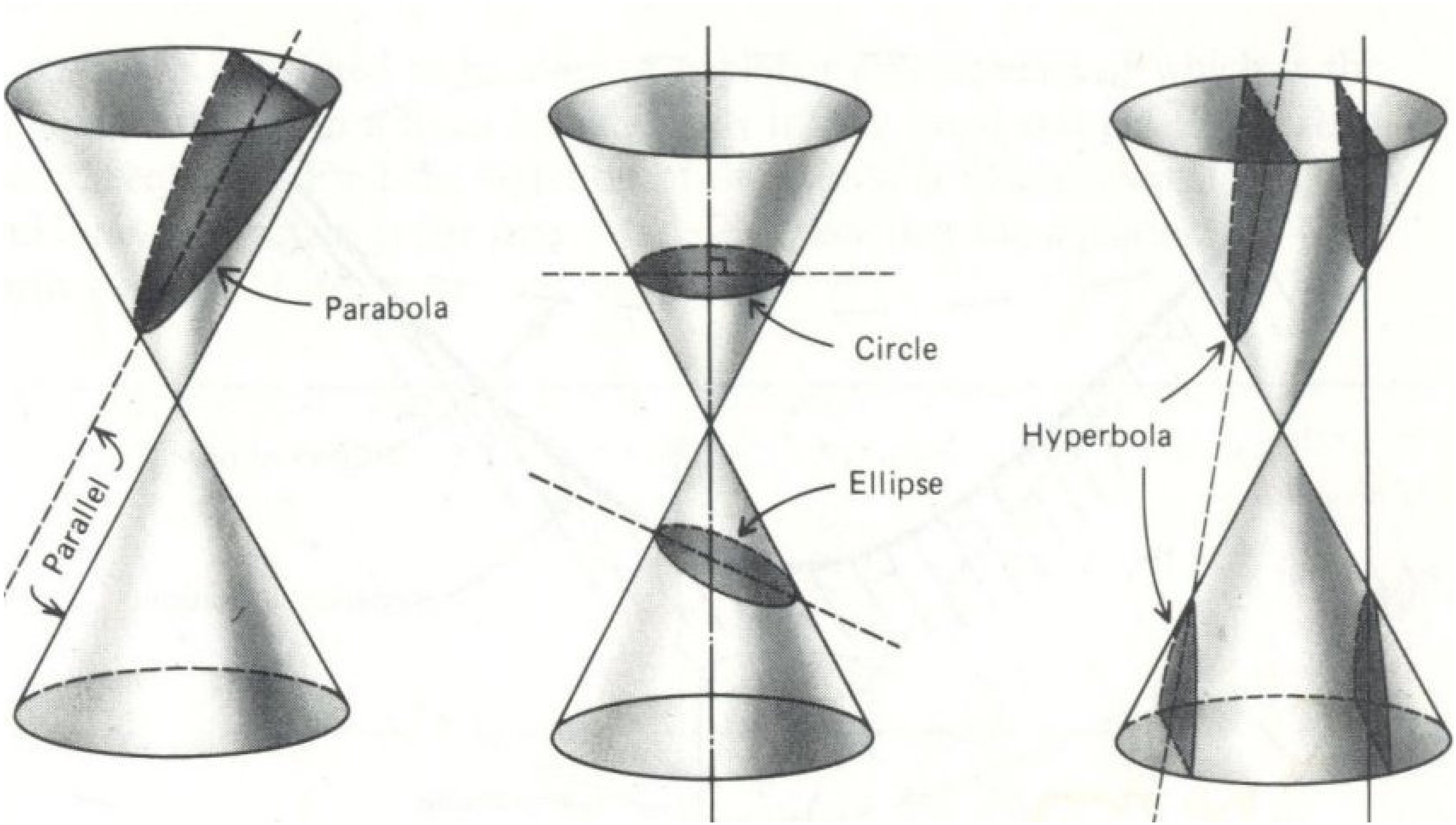
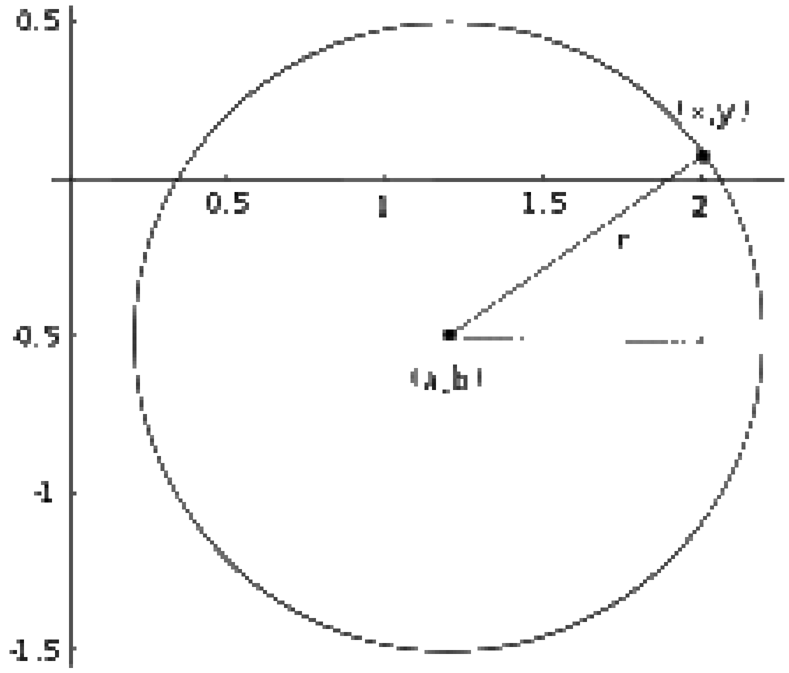


Image from http://www.andrews.edu/~calkins/math/webtexts/geom09.htm

Equation of Circle

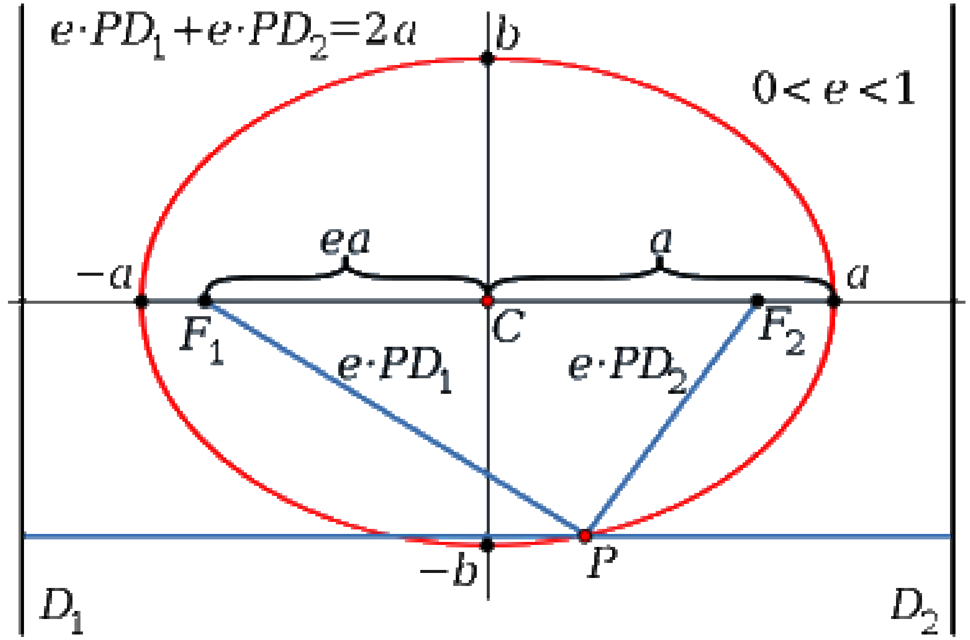


$$(x - a)^2 + (y - b)^2 = r^2$$

Centre of circle (radius r) is (a,b)

Image from wikipedia

Equation of Ellipse



$(x/a)^2 + (y/b)^2 = 1$

$y = 0 \quad x = +/- a$

$x = 0 \quad y = +/- b$

$$e = \varepsilon = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

Image from wikipedia

Equation of Ellipse

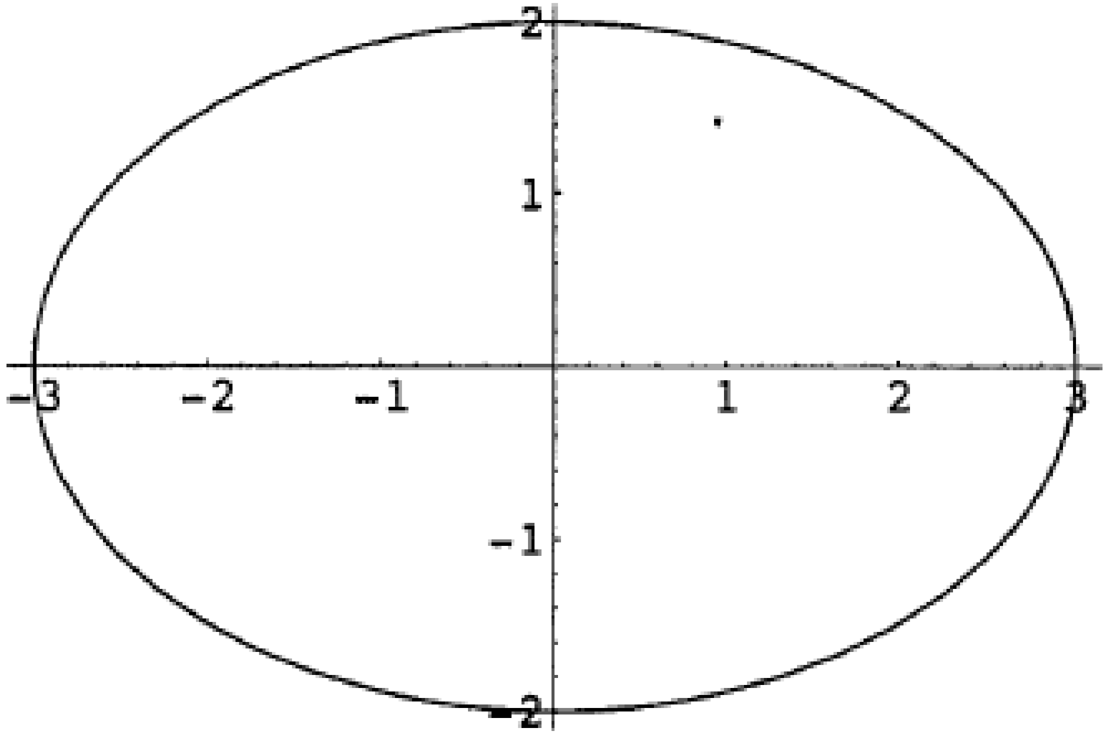


Figure 3: $\frac{x'^2}{9} + \frac{y'^2}{4} = 1$

Image from Barling et al. “Engineering Mathematics 1 Student Notes”, 2009

Equation of Hyperbola

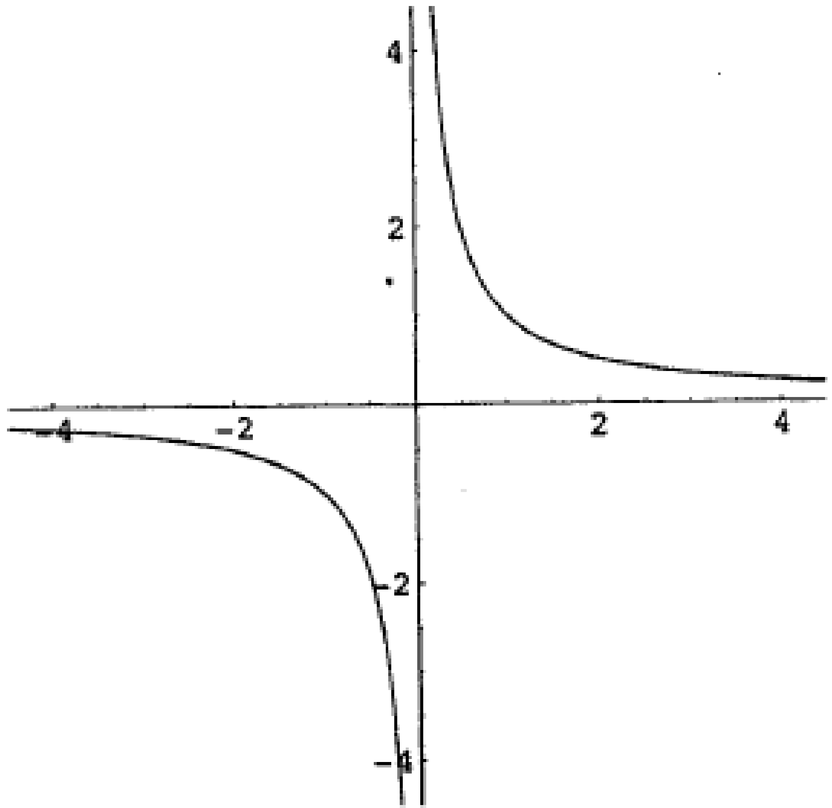


Figure 4: $xy = 1$

Image from Barling et al. “Engineering Mathematics 1 Student Notes”, 2009

Equation of Hyperbola

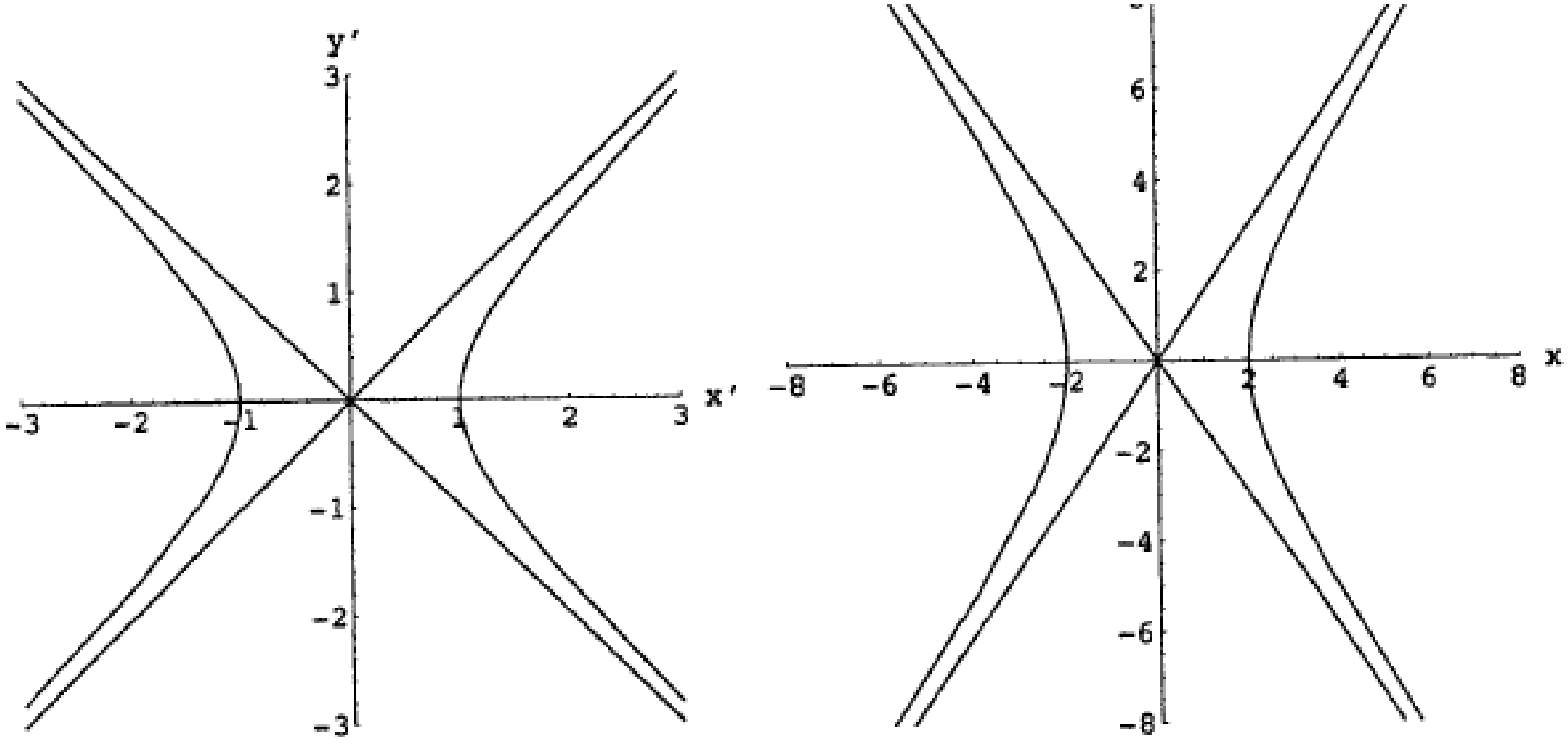


Figure 5: The hyperbola $x'^2 - y'^2 = 1$ and the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Image from Barling et al. "Engineering Mathematics 1 Student Notes", 2009

Equation of Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Such a hyperbola may be sketched by first finding the x-intercepts by putting $y = 0$, this gives $\frac{x^2}{a^2} = 1$ so $x^2 = a^2$ and the x-intercepts are $x = a$ and $x = -a$. Now we can re-arrange the equation to the hyperbola to $\frac{y^2}{b^2} = (\frac{x^2}{a^2} - 1)$. From this we get $y^2 = b^2(\frac{x^2}{a^2} - 1)$ or

$y^2 = \frac{x^2b^2}{a^2}(1 - \frac{a^2}{x^2})$ so we have

$y = \pm\frac{xb}{a}\sqrt{1 - \frac{a^2}{x^2}}$.

From this equation we see that for large values of x the term $\frac{a^2}{x^2}$ is very small so that for large values of x the value of the expression $\sqrt{1 - \frac{a^2}{x^2}}$ approaches 1 and $y = \pm\frac{bx}{a}$ for large values of x. That is the asymptotes to the hyperbola are the straight lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$. This can be seen in the same way as the previous one.

Image from Barling et al. “Engineering Mathematics 1 Student Notes”, 2009

Equation of Hyperbola

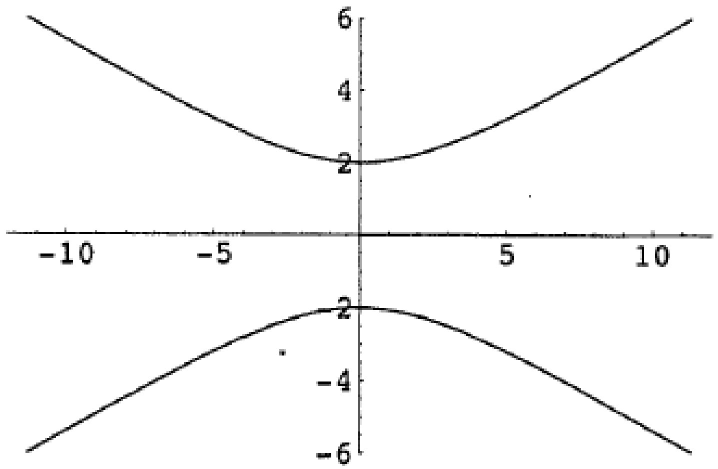


Figure 7: Graph of $\frac{y^2}{4} - \frac{x^2}{16} = 1$

A hyperbola whose equation is of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

has no x-intercepts, but has y-intercepts $\pm b$. (This can be seen by putting $x = 0$.) The asymptotes are the straight lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$

Image from Barling et al. "Engineering Mathematics 1 Student Notes", 2009

Quadratic Functions

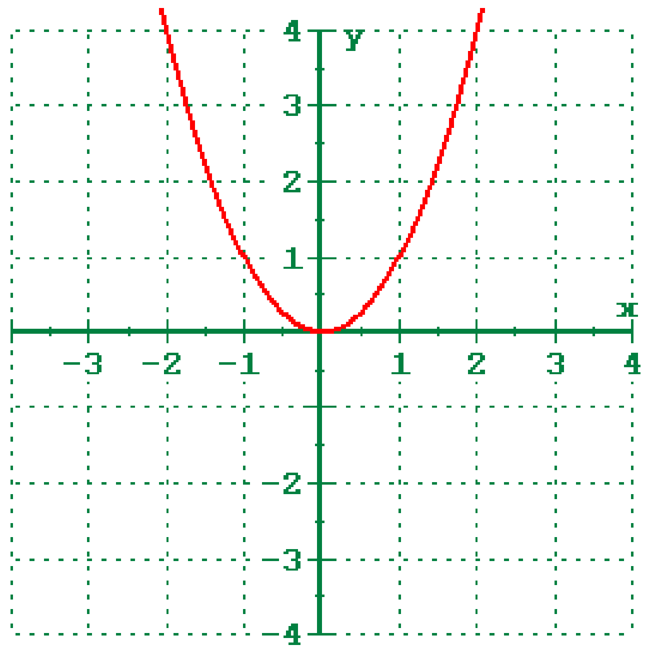
$$y = ax^2 + bx + c$$

$$= a(x+ b/2a)^2 + (c - b^2/2a)$$

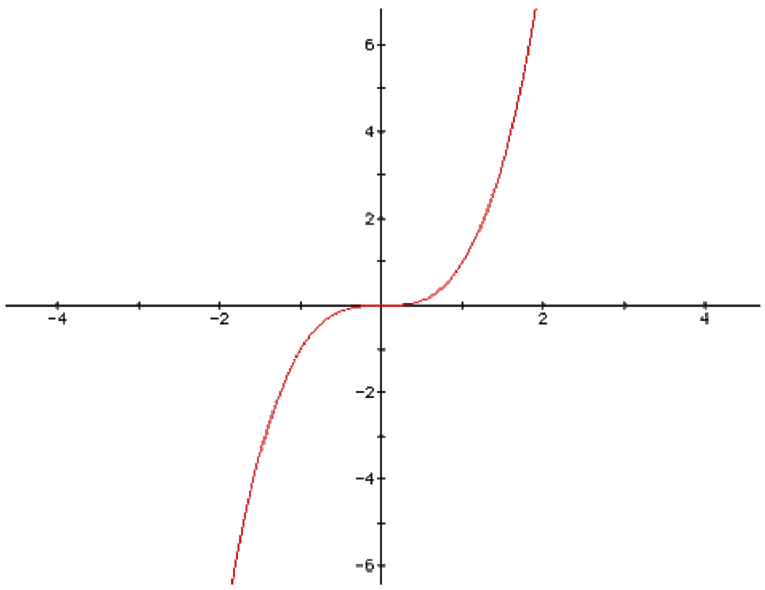
horizontal shift of vertex (-b/2a)

vertical shift of vertex $(c - b^2/2a)$

“a” dilates the shape of the quadratic



Cubic Functions



Simplest Form

$y= x^3$

Related form

$y = a(x-h)^3 + c$

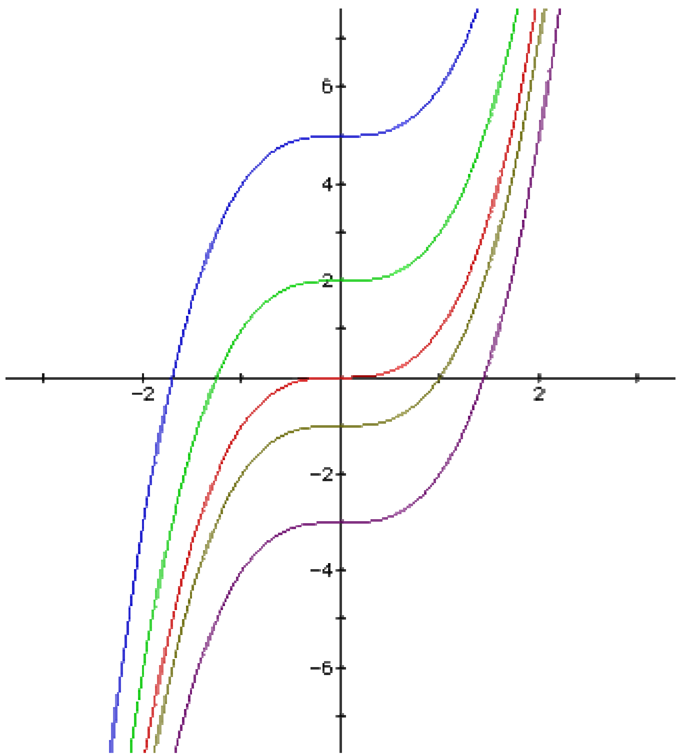
"h" represents a shift of the "x^3" to the right

"c" represent a vertical shift of "x^3"

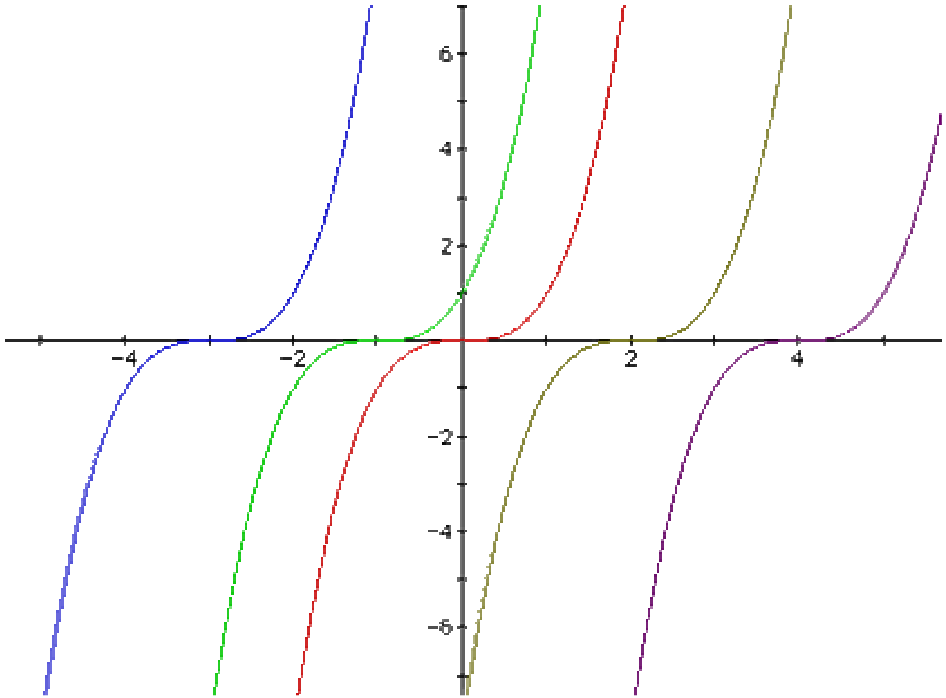
"a" represents the dilation

Image from http://jwilson.coe.uga.edu/

Simple Cubic Curves



Vertical Shift (c)



Horizontal Shift (h)

General Cubic Functions

$y = ax^3 + bx^2 + cx + d$

y intercept is “d”

x intercepts from solving roots of the equations

maxima and minima from $dy/dx = 0$

A GRAPHICS CALCULATOR WOULD BE HANDY !

General Cubic Functions

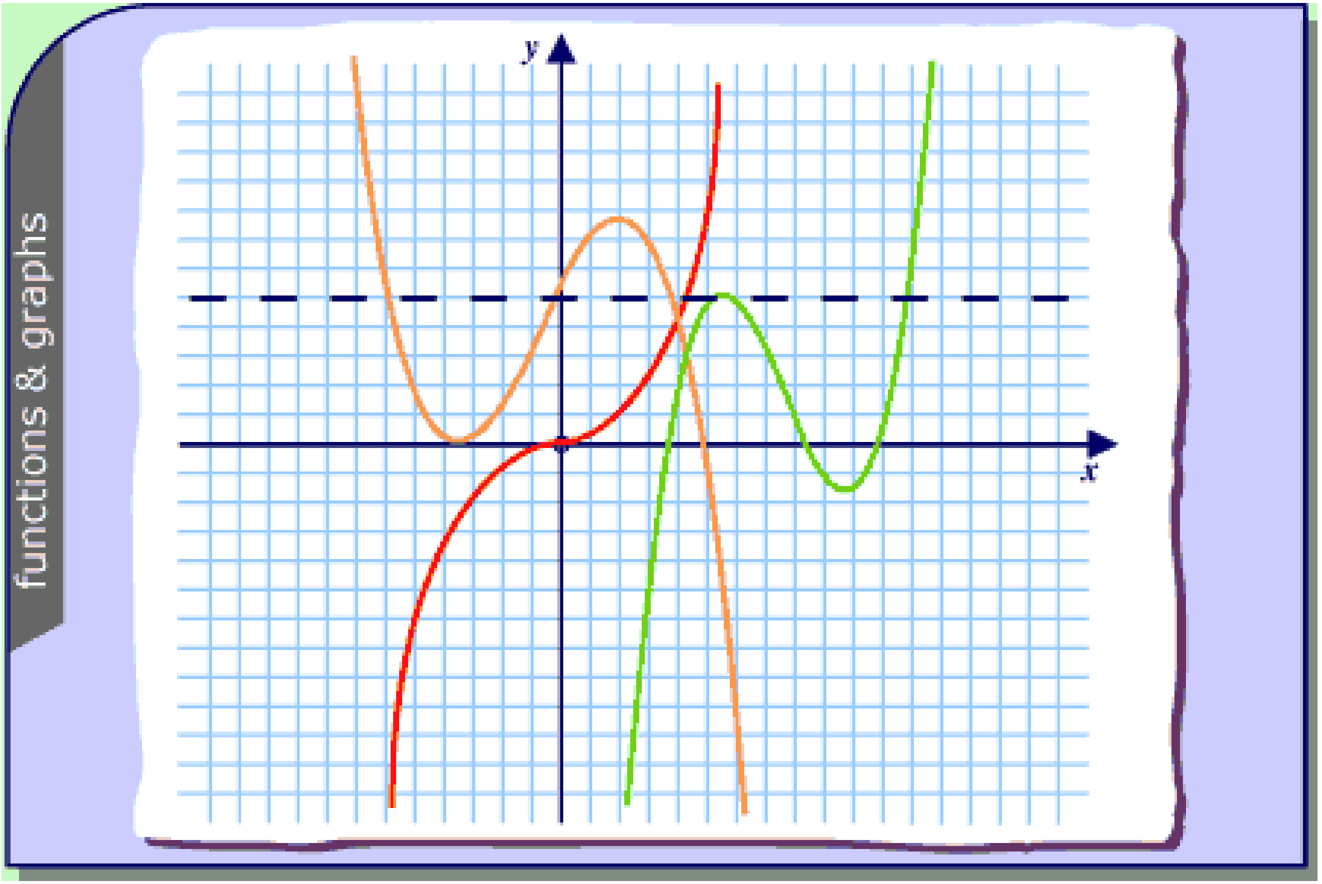


Image from bbc.co.uk

Polynominial Functions

$$y = a_n x^n \ldots. + a_2 x^2 + a_1 x + a_o$$

Same general approach as cubics (roots, derivatives, y intercept)

Negative "n" means that large values of x result in the opposite sign of y for large negative values of x

Negative "n" means that large positive and negative values of x result in the same sign of y

Rational Functions

R(x) = ratio of two polynominials

e.g. R(x) $=1/x$

$= (x^2-5x+7)/(3x^3-5x^2+4x+8)$

etc

Rectangular Hyperbola

Form $y = a + (b/(x-c))$

Vertical asymptote is $x = c$

Horizontal asymptote is $y = a$

Turn $y = (dx + e)/(fx + g) = a + (b/(x-c))$

Rectangular Hyperbola

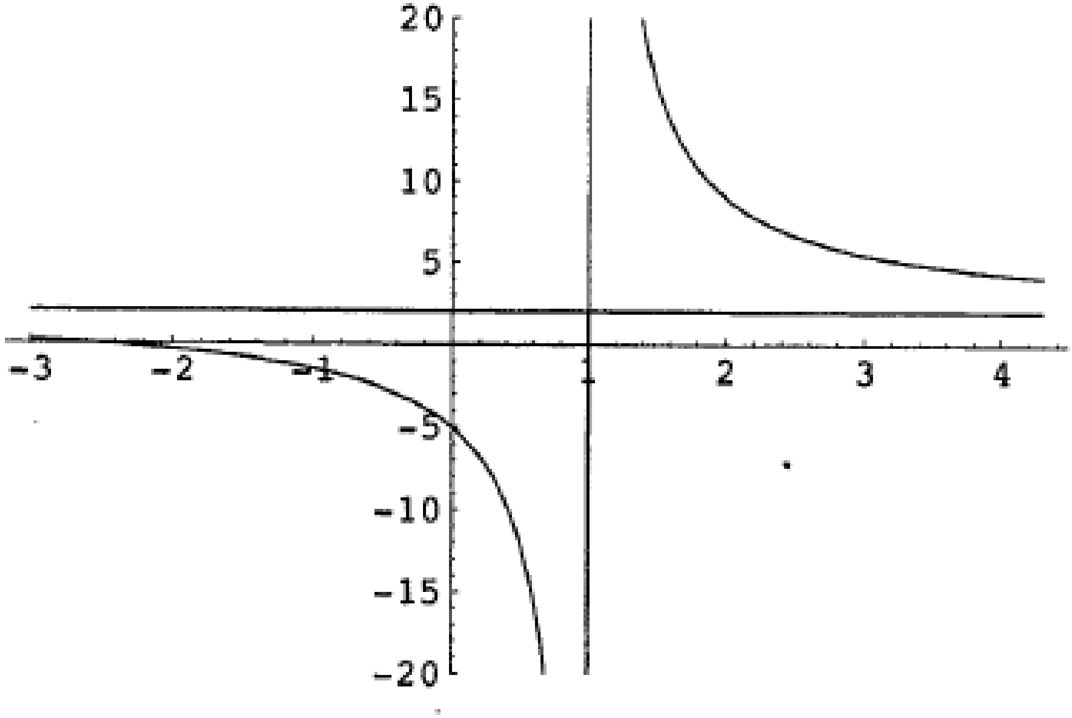


Figure 12: Graph of $y = \dfrac{2x+5}{x-1}$ and $y = 2$.

Image from Barling et al. “Engineering Mathematics 1 Student Notes”, 2009

Higher Order

$y = (ex^2 + f)/(gx + h) = ax + b + (c/(x-d))$

In this relationship the line $(ax + b)$ is a sloping asymptote and d is the vertical asymptote

Higher Order

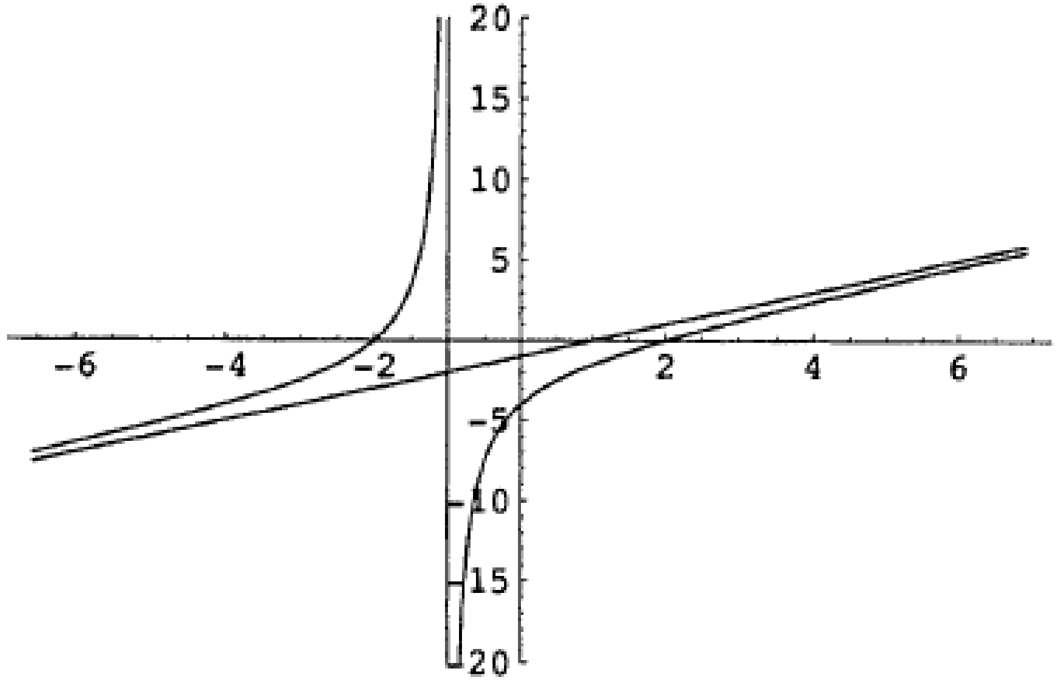


Figure 13: Graph of $y = \dfrac{x^2 - 4}{x + 1}$ and asymptote $y = x - 1$.

Image from Barling et al. “Engineering Mathematics 1 Student Notes”, 2009

Rational Functions

Example 2 Express

$$\frac{x^2-4}{(x-1)^2(x+5)}$$

in partial fractions

Solution We assume

$$\frac{x^2-4}{(x-1)^2(x+5)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+5}$$

Now multiply both sides of this equation by $(x-1)^2(x+5)$. This gives, after cancelling,

$$x^2 - 4 = A(x-1)(x+5) + B(x+5) + C(x-1)^2 \qquad (5)$$

We can first determine B and C. Put $x = 1$ then we have

$1^2 - 4 = A(1-1)(1+5) + B(1+5) + C(1-1)^2$

so

$-3 = 6B$, that is $B = -\frac{1}{2}$. Now put $x = -5$ and we have

$(-5)^2 - 4 = A(-5-1)(-5+5) + B(-5+5) + C(-5-1)^2$

$21 = 0 + 0 + 36C$

so $C = \frac{7}{12}$. To determine A expand the right hand side of 5. This gives

$$x^2 - 4 = (A+C)x^2 + (2A + B - 2C)x + (3B + C - 3A)$$

Equating co-efficients of x^2 on both sides gives $1 = A + C$, and since $C = \frac{7}{12}$ we have $A = 1 - \frac{7}{12} = \frac{5}{12}$ and

$$\frac{x^2-4}{(x-1)^2(x+5)} = \frac{5/12}{x-1} - \frac{1/2}{(x-1)^2} + \frac{7/12}{x+5}.$$

Image from Barling et al. "Engineering Mathematics 1 Student Notes", 2009

Rational Functions

Example 3 Express

$$\frac{x^2 - 3x - 8}{(x^2 + 4x + 5)(x + 1)}$$

in partial fractions

Solution Note that we cannot factorize $x^2 + 4x + 5$ so we assume

$$\frac{x^2 - 3x - 8}{(x^2 + 4x + 5)(x + 1)} = \frac{Ax + B}{x^2 + 4x + 5} + \frac{C}{x + 1}$$

Now multiply both sides of this equation by $(x^2 + 4x + 5)(x + 1)$. This gives, after cancelling,

$$x^2 - 3x - 8 = (Ax + B)(x + 1) + C(x^2 + 4x + 5) \qquad (6)$$

We can first determine C by putting $x = -1$ then we have

$(-1)^2 - 3(-1) - 8 = (A(-1) + B)(-1 + 1) + C((-1)^2 + 4(-1) + 5)$

so

$1 + 3 - 8 = C(1 - 4 + 5)$, that is $-4 = 2C$, so $C = -2$. Now expanding 6 gives

$$x^2 - 3x - 8 = (A + C)x^2 + (A + B + 4C)x + (B + 5C)$$

Equating co-efficients of x^2 on both sides gives $1 = A + C$, and since $C = -2$ we have $A = 1 - (-2) = 3$. To determine B equate constant terms on both sides of the equation. This gives $-8 = B + 5C$ and since $C = -2$ we have $-8 = B + 5(-2)$ so $B = -8 + 10 = 2$ so we have

$$\frac{x^2 - 3x - 8}{(x^2 + 4x + 5)(x + 1)} = \frac{3x + 2}{x^2 + 4x + 5} - \frac{2}{x + 1}.$$

Image from Barling et al. “Engineering Mathematics 1 Student Notes”, 2009

Rational Functions R(x) = P(x)/Q(x)

- To a factor $x - a$ of $Q(x)$ there corresponds a partial fraction of the form $\dfrac{A}{(x-a)}$
- To a repeated factor $(x-a)^2$ of $Q(x)$ there corresponds two partial fractions of the form $\dfrac{A_1}{(x-a)} + \dfrac{A_2}{(x-a)^2}$
- In general to a repeated factor $(x-a)^m$ of $Q(x)$ there corresponds a group of partial fractions of the form $\dfrac{A_1}{(x-a)} + \cdots \dfrac{A_m}{(x-a)^m}$
- To a quadratic factor $x^2 + px + q$ of $Q(x)$ there corresponds a partial fraction of the form $\dfrac{Bx + C}{x^2 + px + q}$
- To a repeated quadratic factor $(x^2 + px + q)^m$ of $Q(x)$ there corresponds a group of partial fractions of the form $\dfrac{B_1x + C_1}{x^2 + px + q} + \cdots + \dfrac{B_mx + C_m}{(x^2 + px + q)^m}$

Image from Barling et al. “Engineering Mathematics 1 Student Notes”, 2009

Transcendental Functions

First identified by Leibniz (1646-1716) – lawyer, chemist, philosopher, mathematician and Newton's rival

Invented calculus and binary mathematics in his spare time !

Transcendental numbers and functions = numbers functions that cannot be obtained by algebraic equations

Is □ transcendental ?

Is $(2)^{0.5}$ transcendental ?

Exponential Functions

$a^m = a \times a \times a \times a \ldots.a$ m times

a = "base" number and n = "exponent" or the "power"

Index Laws

$a^m a^n = a^{m+n}$

$a^n/a^m = a^{n-m}$ (assuming n>m)

$(a^m)^n = a^{mn}$

Exponential Functions

Functions that grow or decay at a constant rate

Population growth and compound interest follow:

$$P(n) = P_o(1 + (\%r/100))^n$$

Very common in science and engineering

“the exponential function”

$y = e^x$

Always positive

$dy/dx = e^x$

e = transcendental number

= 2.71828182 ….

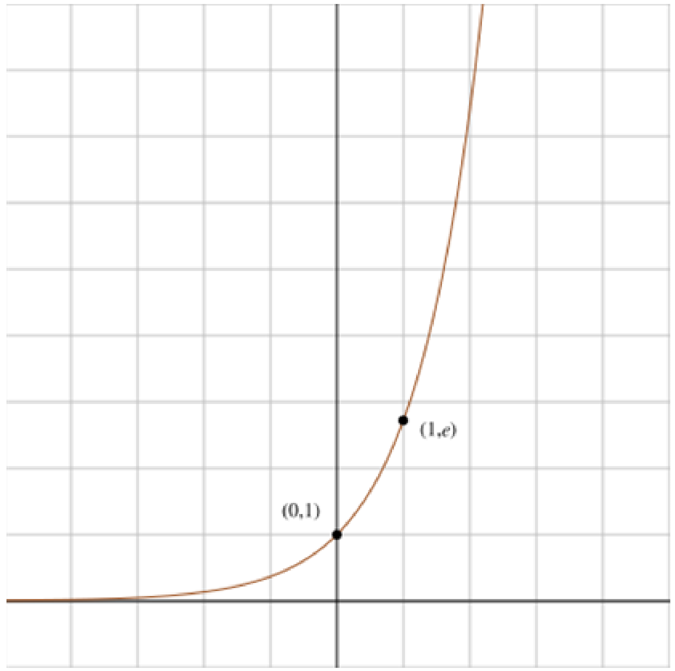
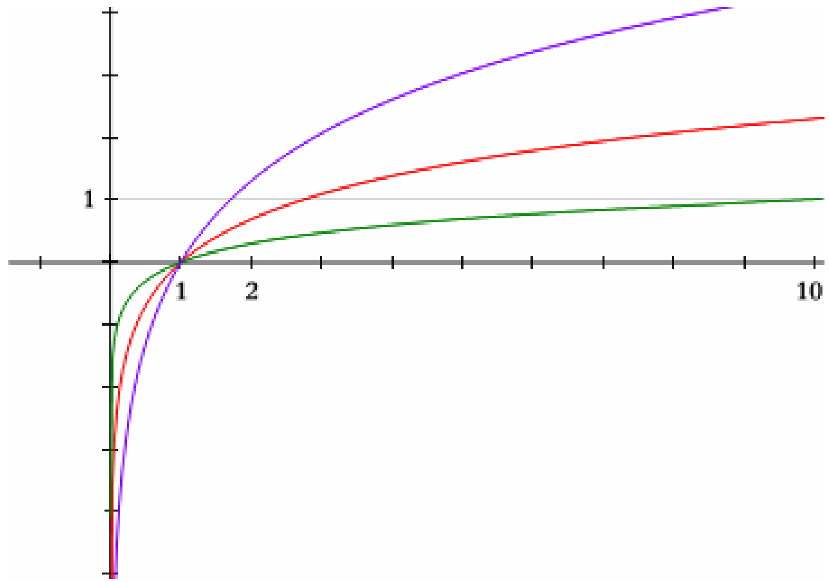


Image from wikipedia

Logarithmic functions

$y = \log_a x$ if and only if $x = a^y$

$y = \ln x$ if an only if $x = e^y$



Logarithms to various bases: red is to base e, green is to base 10, and purple is to base 1.7.

Laws of logarithms

$\log_a (AB) = \log_a(A) + \log_a(B)$

$\log_a(A/B) = \log_a(A) – \log_a(B)$

$\text{loga}(A^n) = n\log_a(A)$

Image from wikipedia

Trigonometric Functions

Angles can be measured by:

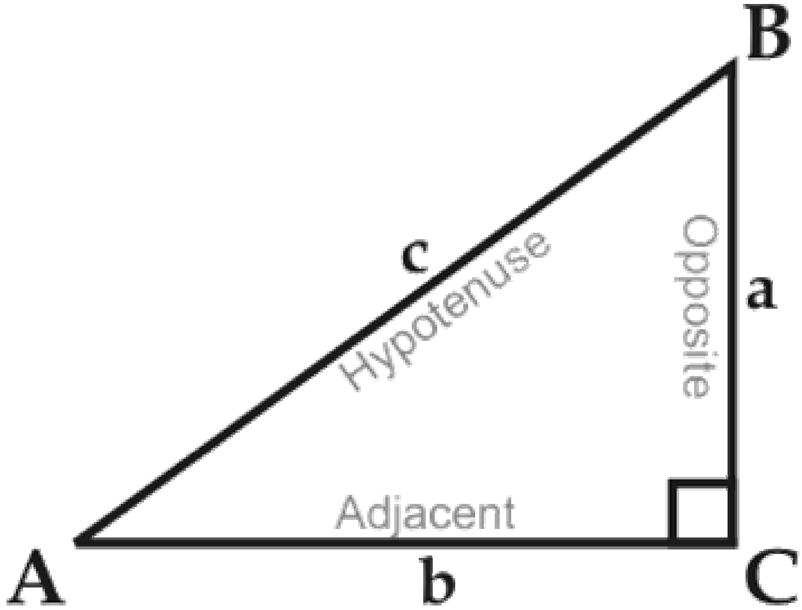
Degrees – 1/360 of a circle = 1º

Radian – 360/2□ = 1 radian = 1^c = 57.295779º

180º = □, 90º = □/2 , 60º = □/3 , 45º = □/4

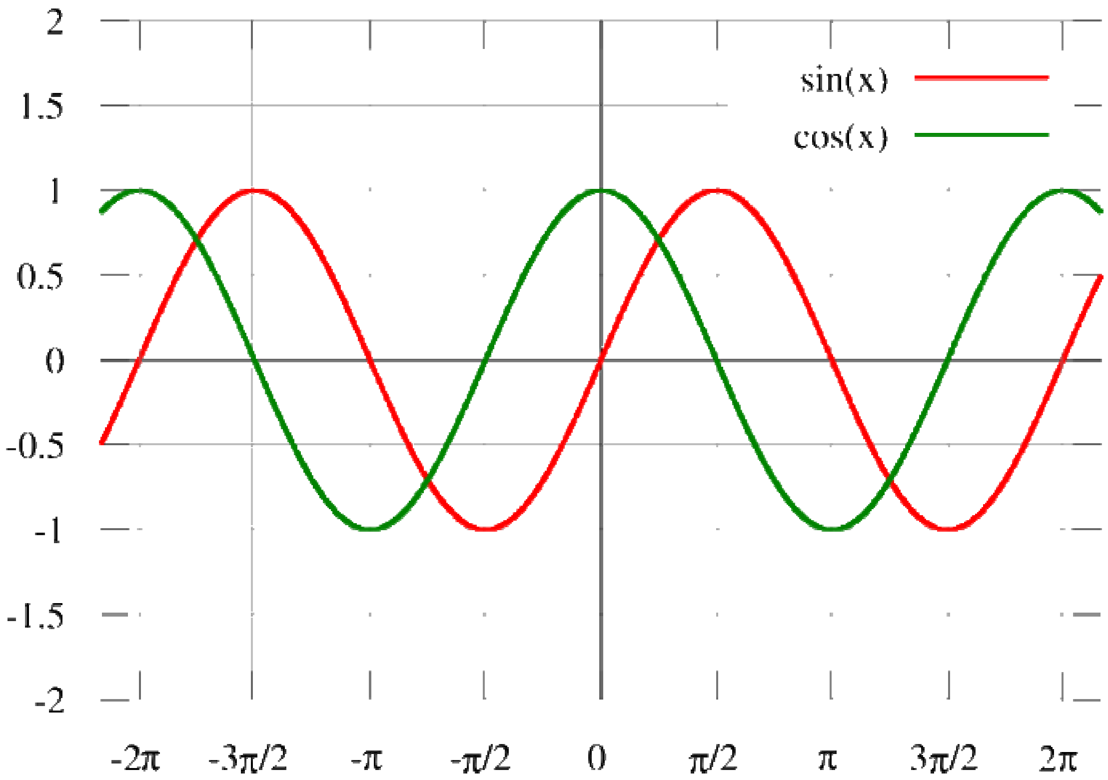
Image from Wikipedia

SOHCAHTOA !!!!



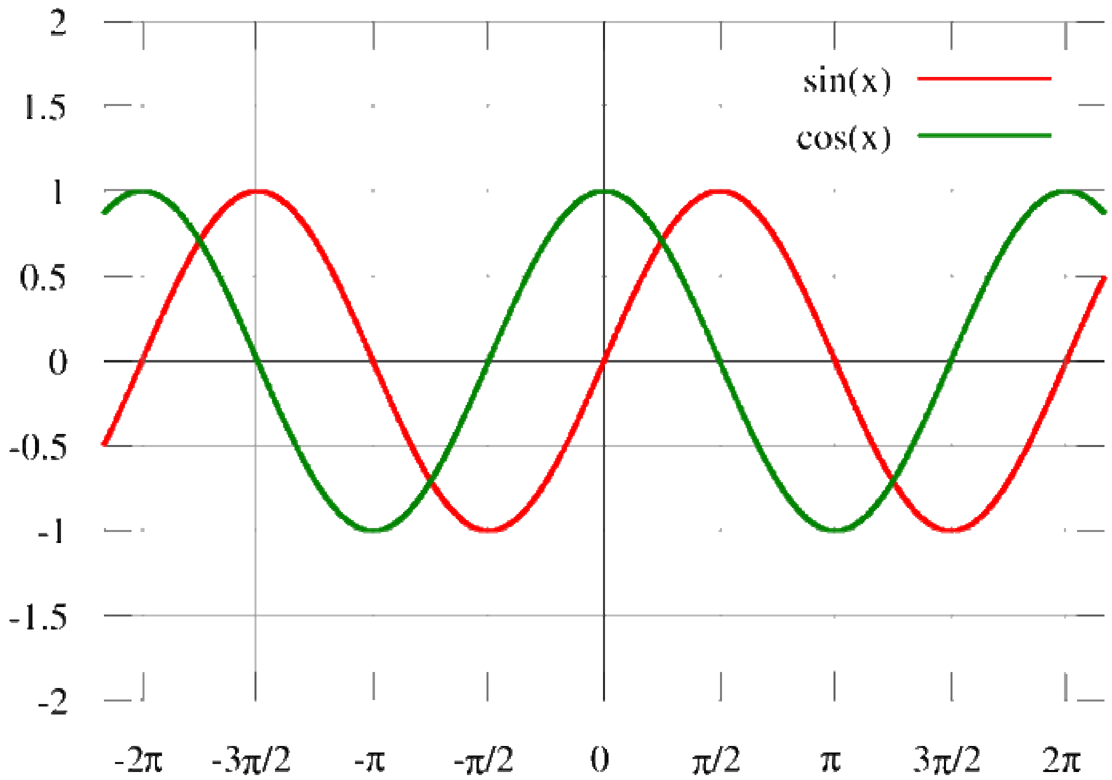
$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin A}{\cos A}$$

Image from Wikipedia



cos 0 = 1, sin 0 =1, cos /2 = 0, sin □/2 = 1, cos □/4 = sin □/4 = $2^{0.5}/2$,
cos □/3 = sin □/6 =1/2

Image from Wikipedia



cos (❑ - ❑/2) = sin ❑

cos (❑ + ❑/2) = -sin ❑

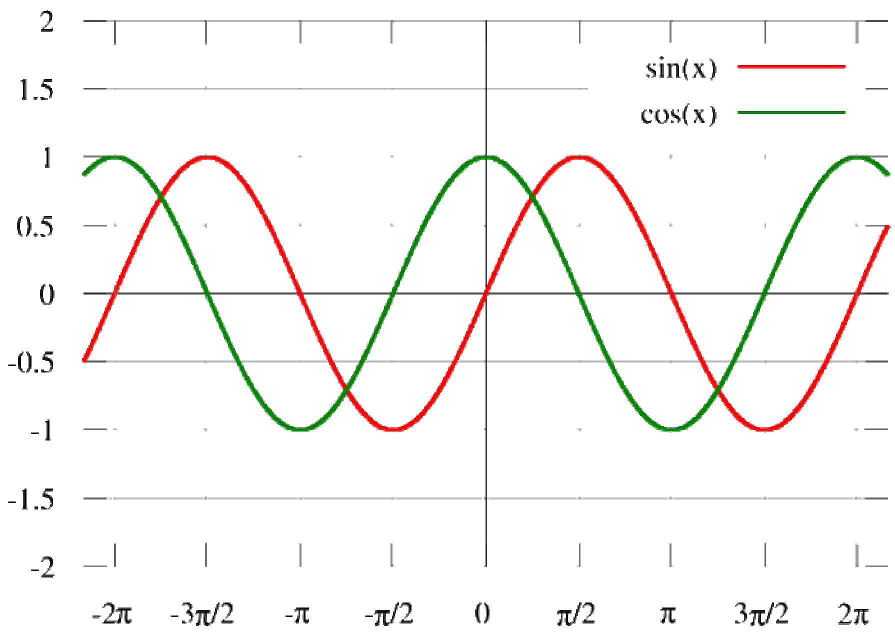
sin (❑ + 2k❑) = sin ❑

cos (❑ + 2k❑) = cos ❑

Image from Wikipedia

k= integer

Quadrants



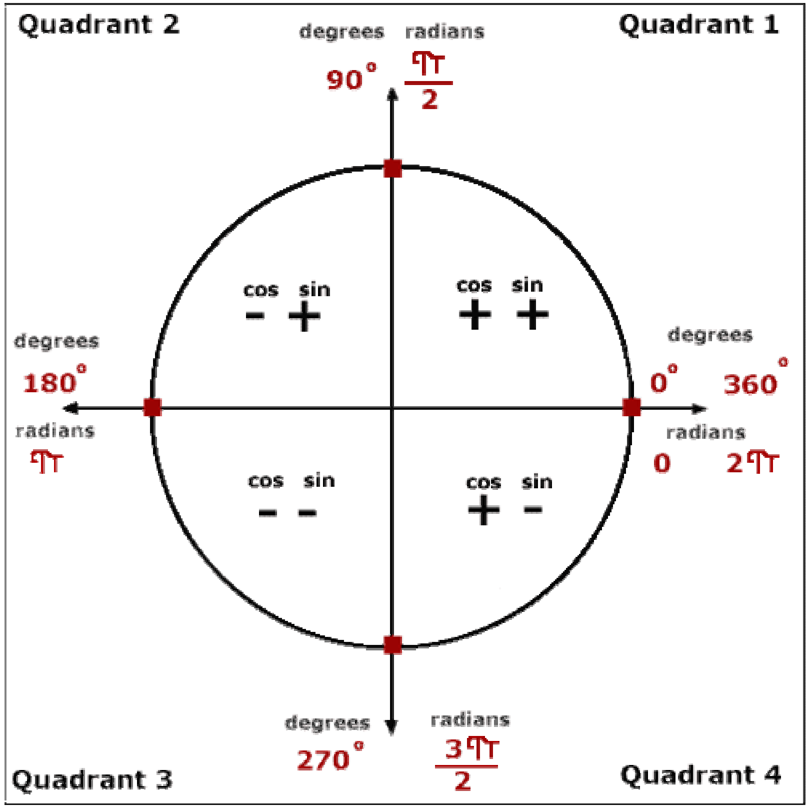


Image from www.marsenlite.com

Quadrants

Unit Circle

$\cos^2\theta + \sin^2\theta = 1$

$\sin\theta = +/- (1 - \cos^2\theta)^{0.5}$

Need additional information to locate quadrant

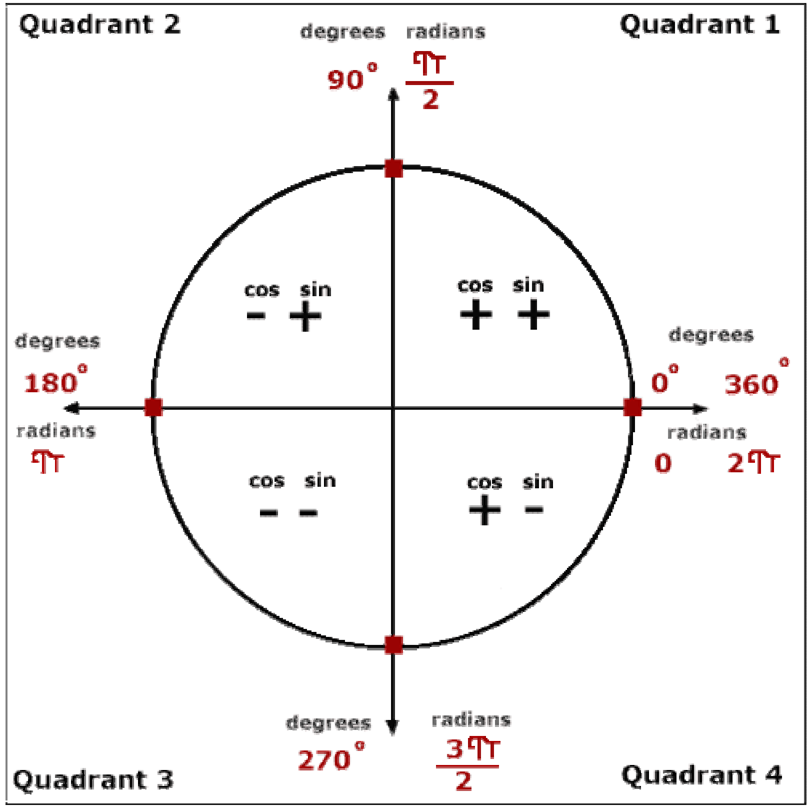


Image from www.marsenlite.com

Relationships between Sin and Cos

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Double Angle Formulae

Using $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos^2 A + \sin^2 A = 1$

We can easily derive

$$\cos(2A) = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

Double Angle Formulae

Using cos(A - B) = cos A cos B + sin A sin B

$a^2 + b^2 = R^2$

where a, b and R are the right angle triangle with angle B opposite ♋

We can easily derive

a cos ❑ + b sin ❑ = R cos (❑ - ♋)

where tan ♋ = b/a

Other Trigonometric Functions

tan ❑ = sin ❑ / cos ❑

cot ❑ = cos ❑ / sin ❑

cosec ❑ = 1/ sin ❑

sec ❑ = 1/ cos ❑

Graph of Tan ❑

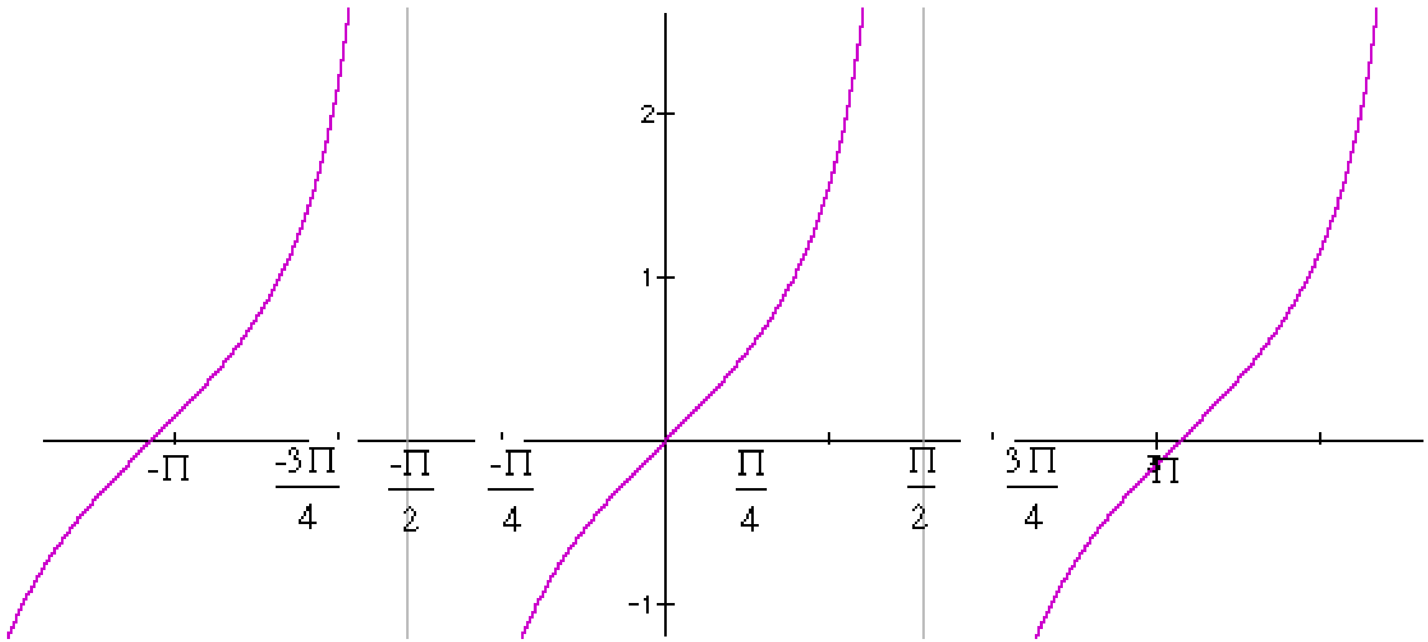


Image from www.acts.tinet.ie/trigon/trigonometry_171.gif

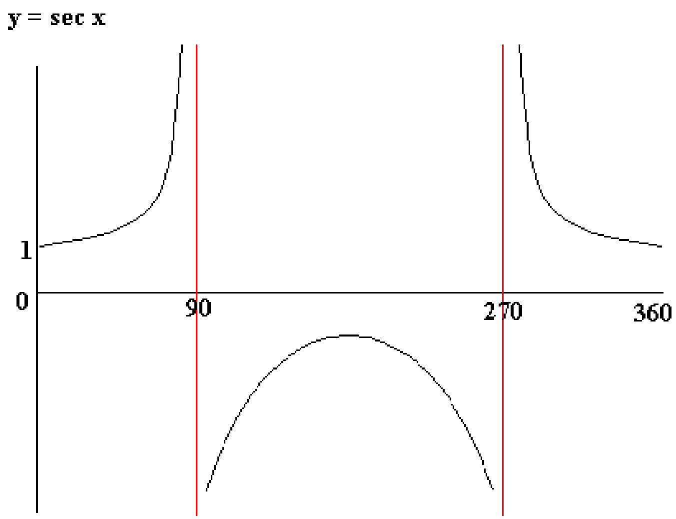
Cosec ❑ and Sec ❑

Note: Unit Circle

cosec ❑ = 1/y

sec ❑ = 1/x

Cot ❑ = x/y



Red lines are asymptotes

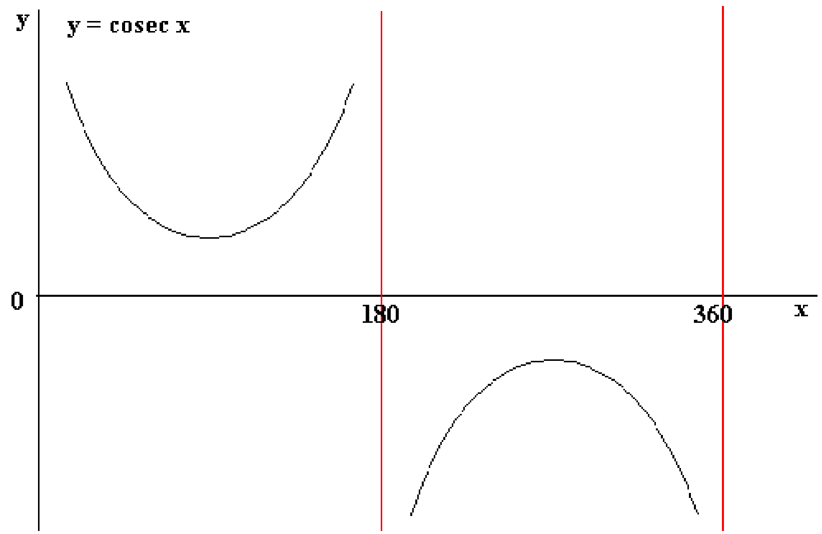
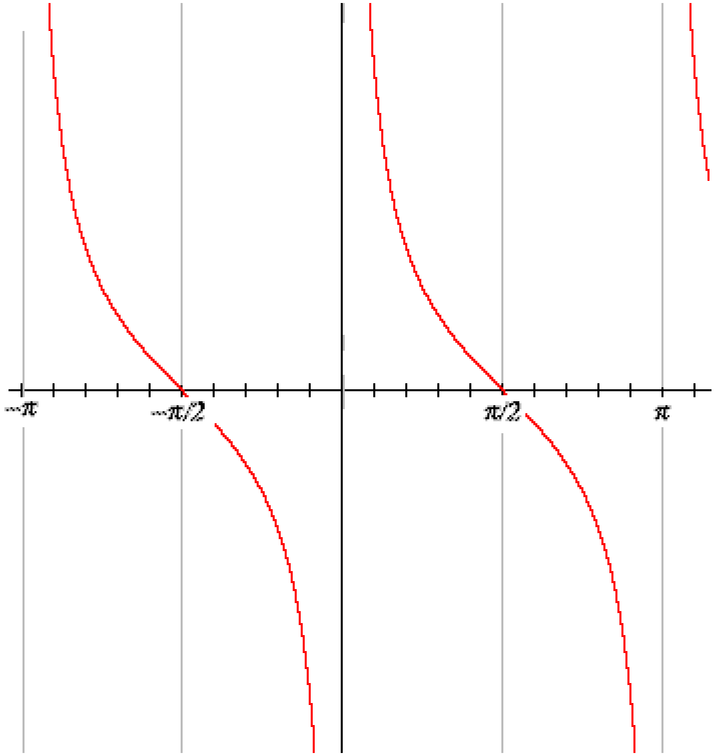


Image from www.mathrevision.com

Cotangent ❑ tan ❑



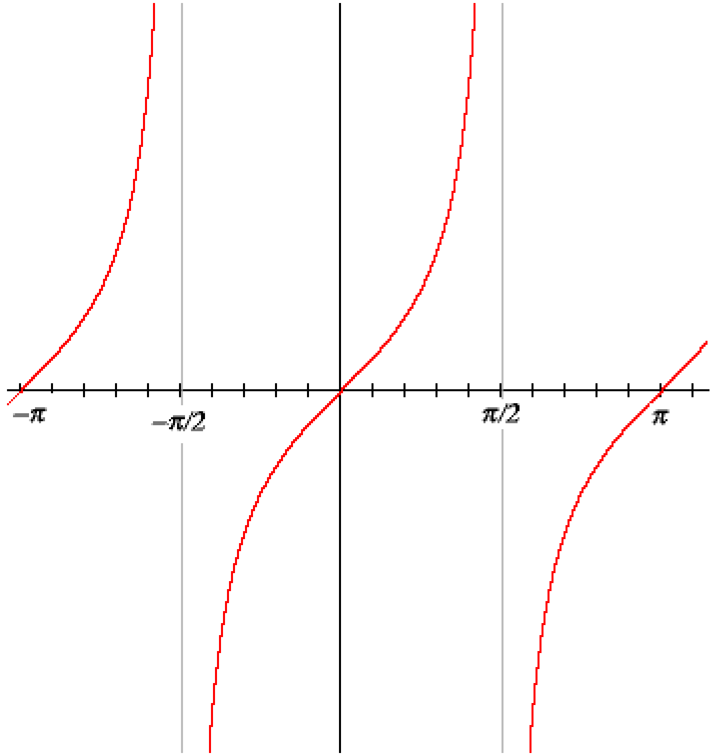


Image from www.clarku.edu/~djoyce/trig/cot.gif

Inverse Trig. Functions

$\sin^{-1} x = \text{arsin } x$ 🕒 cosec x

$\cos^{-1} x = \text{arcos } x$ 🕒 sec x

$\tan^{-1} x = \text{artan } x$ 🕒 cot x

e.g. $\sin(□/4) = 2^{1/2}/2$

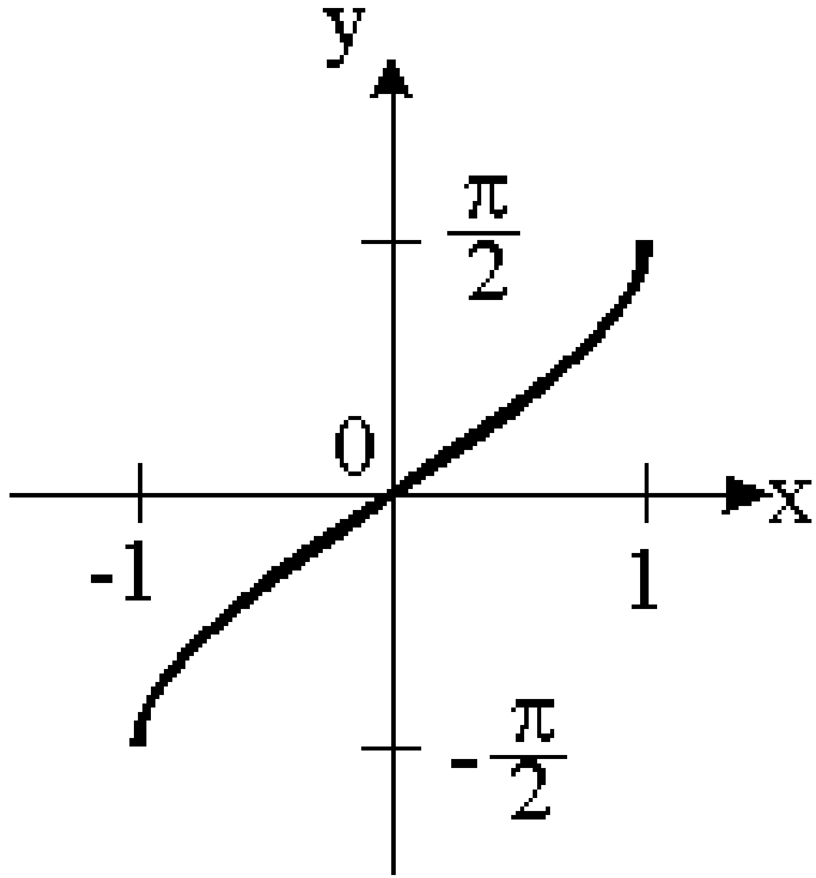
sin-1 $(2^{1/2}/2) = □/4$

Inverse Functions

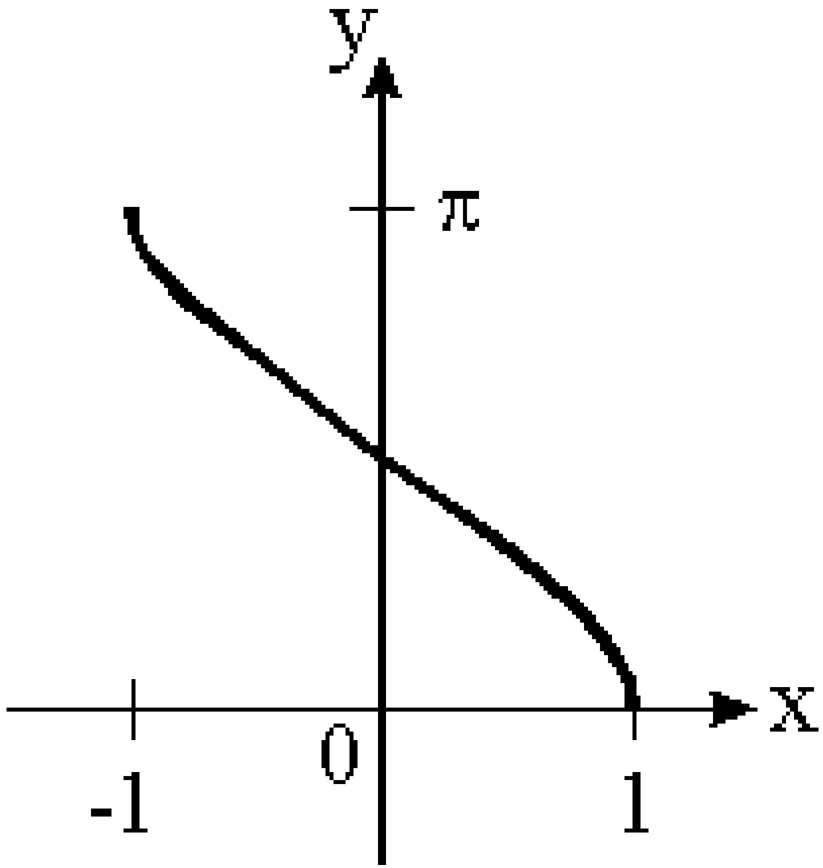
Domain of $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$?

Must maintain symmetry around $y = x$

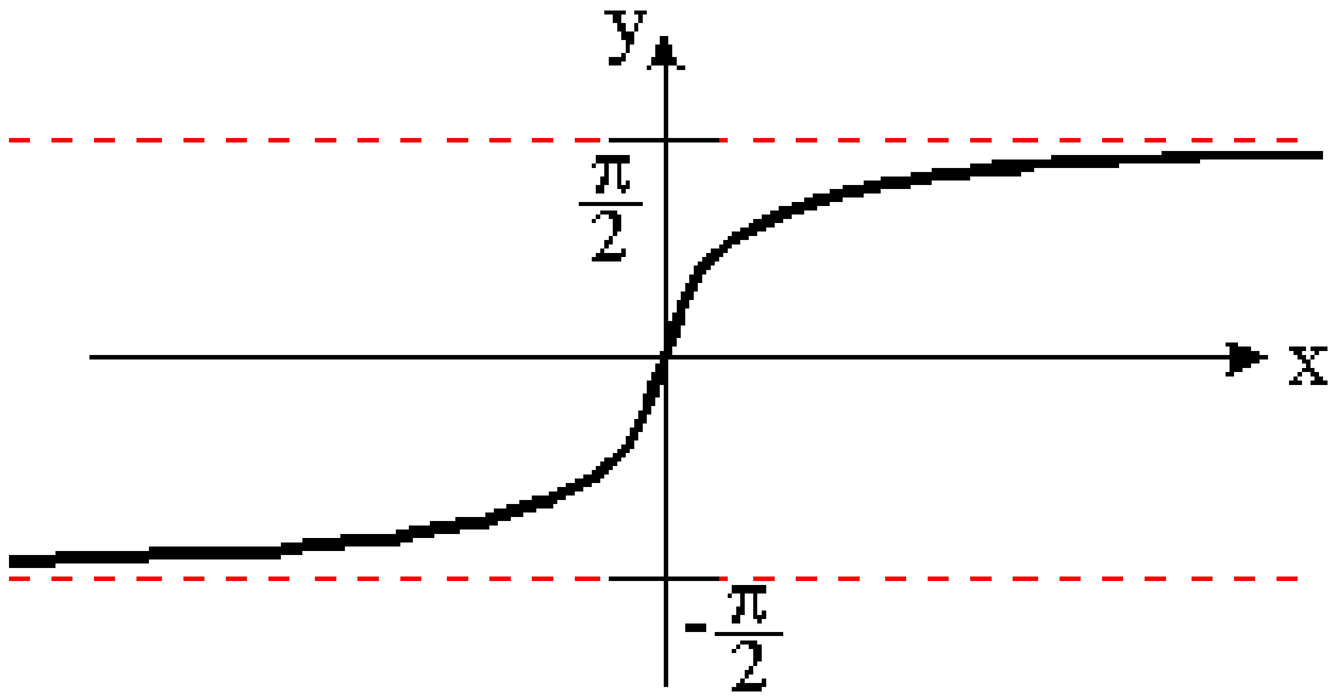
sin x and $\sin^{-1} x$



cos x and cos $^{-1}$ x



tan x and tan $^{-1}$ x



Hyperbolic Functions

Functions analogous to trigonometric functions

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

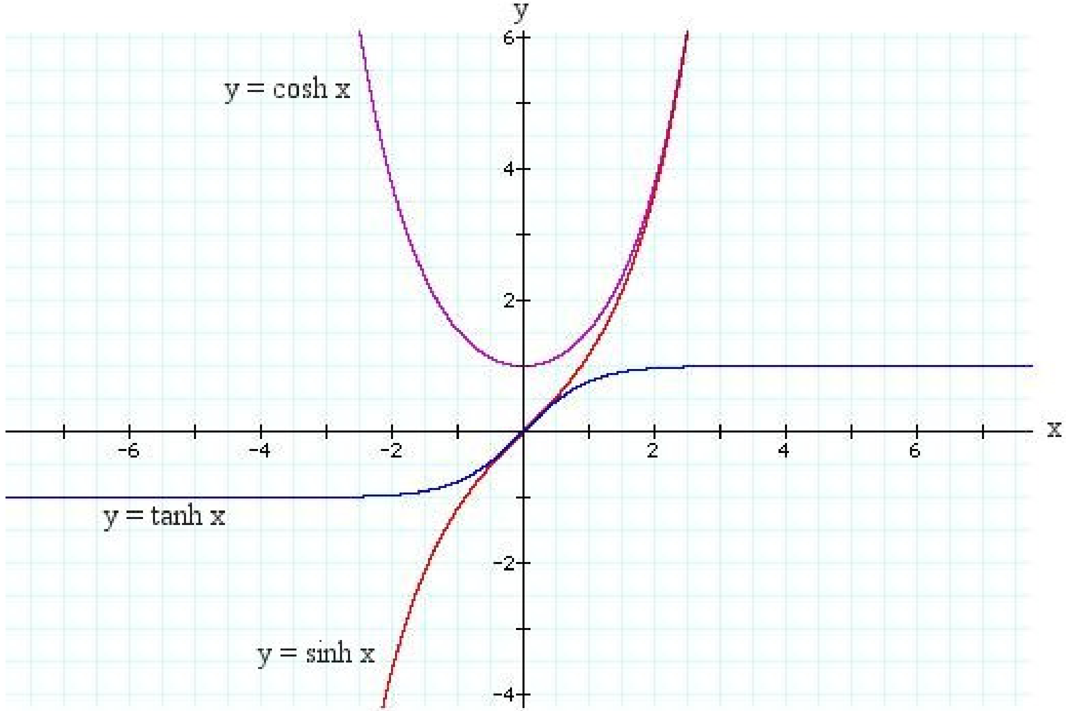


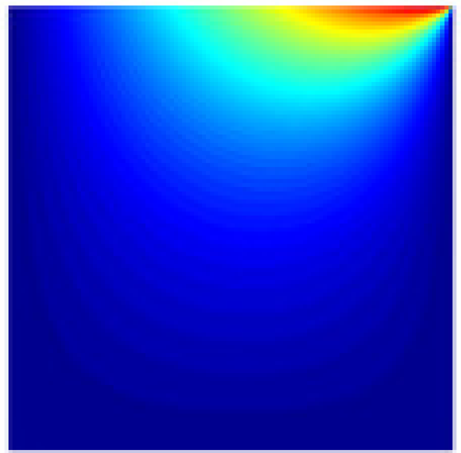
Image from http://universe-review.ca/I15-57-hyperbolic.jpg

Hyperbolic Functions – St Louis Arch

Image from www.robrodriguez.com

Hyperbolic Functions – Heat Transfer

Heat transfer into a plate n meters with co-ordinates x and y



Fourier series expansion of equations results in the following solution for temperature distribution in the plate

$$T(x,y) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \frac{\sinh(ny)}{\sinh(n\pi)}.$$

Image from www.answers.com/topic/fourier-series

Hyperbolic Functions

Important relationships

$\cosh^2 x - \sinh^2 x = 1$

(remember cos^2❑ + sin^2❑ =1)

sechx	$= 1/\cosh x$	$= 2/(e^x + e^{-x})$
cosech x	$= 1/\sinh x$	$= 2/(e^x - e^{-x})$
coth x	$= \cosh x/\sinh x$	$= (e^x + e^{-x}) / (e^x - e^{-x})$

Inverse Hyperbolic Fuctions

$\cosh^{-1}x = \ln\left(x + (x^2-1)^{0.5}\right)$ for x ¤ 1

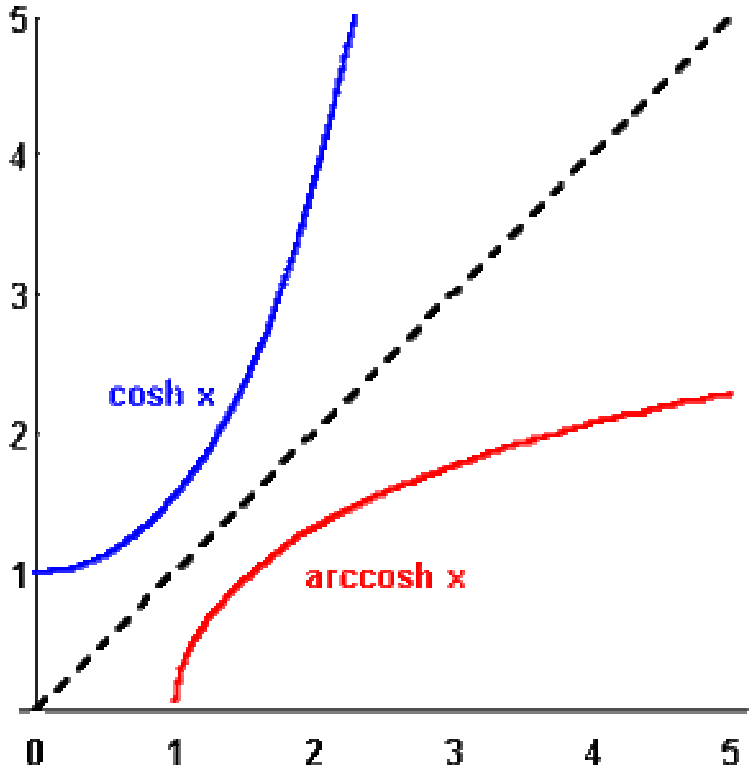


Image from www.sosmath.com/trig/hyper/hyper03/hyper03.html

Inverse Hyperbolic Fuctions

$\sinh^{-1}x = \ln (x + (x^2+1)^{0.5})$ for all x

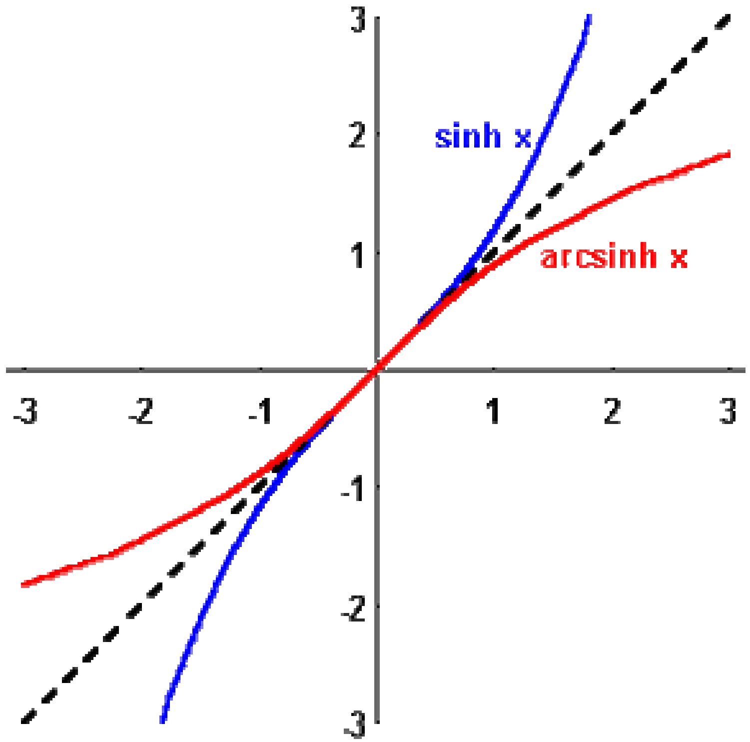


Image from **www.sosmath.com/trig/hyper/hyper03/hyper03.html**

Inverse Hyperbolic Functions

$\tanh^{-1}x = \frac{1}{2} \ln ((1+x)/(1-x))$ defined for $-1 < x < 1$

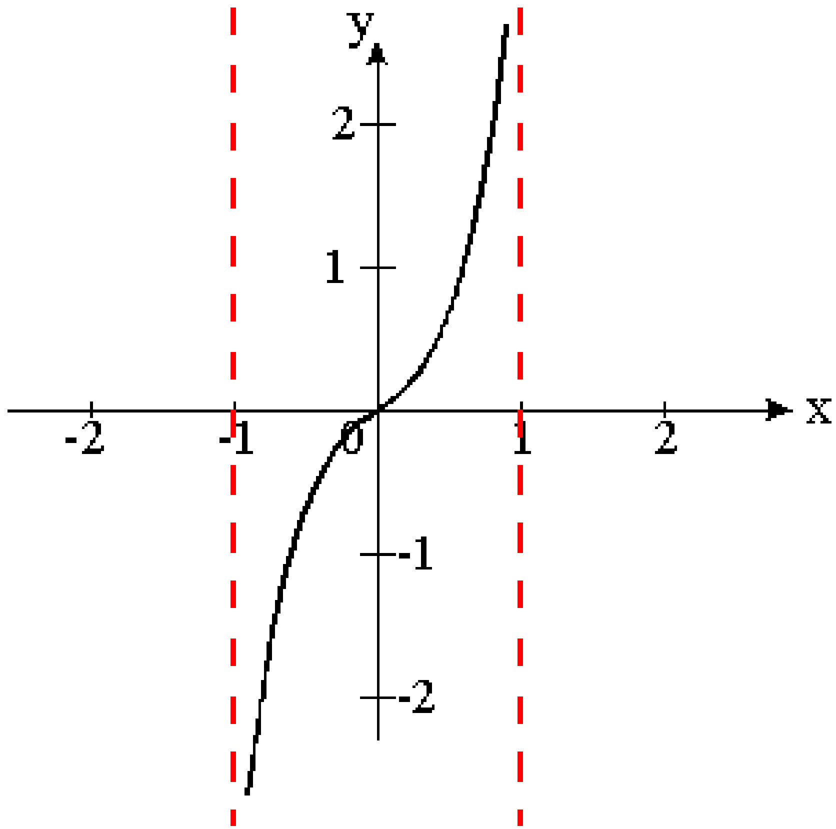


Image from **library.thinkquest.org/10030/7inverse.htm**