Fixed Point Iterative Processes

W.M. Kozlowski

UNSW

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Abstract

Asymptotic Pointwise Mappings

In 2008, Kirk and Xu proved existence of fixed points of asymptotic pointwise nonexpansive mappings $T : C \to C$, *i.e. mappings such that*

$$\|T^n(x) - T^n(y)\| \leq \alpha_n(x)\|x - y\|$$

where $\limsup_{n\to\infty} \alpha_n(x) \leq 1$, *for all* $x, y \in C$, *where* C *is a nonempty, closed, bounded and convex subset of a uniformly convex Banach space* X. *We will discuss the iterative algorithms for the construction of the fixed points of the asymptotic pointwise contractions and asymptotic pointwise nonexpansive mappings and the weak and strong convergence of such algorithms. We will touch on some generalizations of these results to the case of modular functions spaces.*

Contractions

Celebrated Banach Contraction Principle is our starting point even though that the idea of using successive approximations to solve differential and integral equations preceeded it. Note that theorem has three components:

1. Existence of a fixed point
2. Fixed point uniqueness
3. Iterative process for the construction of the unique fixed point by taking the limit of any orbit (i.e. limit of Picard iterates).

Banach Fixed Point Theorem (1922)

Let $C \neq \emptyset$ be a closed subset of a Banach space X, and $T : C \to C$ be a contraction, i.e. there exists a constant $\alpha < 1$ such that

$$\|T(x) - T(y)\| \leq \alpha \|x - y\|$$

for all $x, y \in C$. Then, there exists a unique $z \in C$ such that $T(z) = z$. Moreover, for any $x \in C$, there holds $\|T^n(x) - z\| \to 0$, where T^n is the n-th iterate of T.

Nonexpansive Mappings

The case $\alpha = 1$ waited for a resolution for more than 40 years but then was proved independently by 3 mathematicians. The proof uses weak compactness of closed, bounded sets in uniformly convex (hence reflexive) Banach spaces and is of existential nature by proving the Cauchy property of a sequence of iterates of a point on which a certain type function attains its minimum. Several weak convergence iterative processes were studied for construction of fixed points starting from Opial's paper (1967).

Browder/Göhde/Kirk Fixed Point Theorem (1965)

Let $C \neq \emptyset$ be a closed, covex, bounded subset of a uniformly convex Banach space X (def on slide 18), and $T : C \to C$ be nonexpansive, i.e.

$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all $x, y \in C$. Then, there exists a $z \in C$ such that $T(z) = z$. The set of all fixed points is closed and convex (but can consist of more than one point).

Pointwise Contractions

A generalisation of Banach Contraction Principle to pointwise contractions, where the Lipschitzian constant is replaced by a function. Like Banach Principle, this Fixed Point Theorem has three components:

1. Existence of a fixed point
2. Fixed point uniqueness
3. Iterative construction of the fixed point by taking the limit of an orbit

Belluce (1969)/Kirk(1970) Fixed Point Theorem

Let $C \neq \emptyset$ be a closed, convex, bounded subset of a uniformly convex Banach space, and $T : C \to C$ be a pointwise contraction, i.e. there exists a function $\alpha : C \to [0, 1)$ such that

$$\|T(x) - T(y)\| \leq \alpha(x)\|x - y\|$$

for all $x, y \in C$. Then, there exists a unique $z \in C$ such that $T(z) = z$. Moreover, for any $x \in C$, there holds $\|T^n(x) - z\| \to 0$.

Asymptotic Nonexpansive Mappings - ANM

A generalisation of nonexpansive mappings to the asymptotic case came soon after.

Goebel/Kirk Fixed Point Theorem (1972)

Let $C \neq \emptyset$ be a closed, convex, bounded subset of a uniformly convex Banach space X, and $T : C \to C$ be asymptotic nonexpansive, i.e.

$$\|T^n(x) - T^n(y)\| \leq \alpha_n \|x - y\|$$

for all $x, y \in C$, where $\limsup_{n\to\infty} \alpha_n \leq 1$. Then, there exists a $z \in C$ such that $T(z) = z$. The set of all fixed points is closed and convex.
Note that each $\frac{1}{\alpha_n} T^n$ *is nonexpansive. This important fact allowed to use many methods of the theory of nonexpansive mappings.*

Iterative Fixed Point Construction for Asymptotic Nonexpansive Mappings

Question was asked how to construct fixed points for these mappings. Since foe every T APNM, $\frac{1}{\alpha_n}T^n$ are nonexpansive, it follows that some algorithms, similar to those developed for nonexpansive mappings, could be used. Two most famous and most successful were:

1. Mann process: $x_{k+1} = t_k T^{n_k}(x_k) + (1 - t_k)x_k$;
2. Ishikawa process:
   $x_{k+1} = t_k T^{n_k}(s_k T^{n_k}(x_k) + (1 - s_k)x_k) + (1 - t_k)x_k$.

Schu (1992), Tan and Xu (1993-1994)

Let $C \neq \emptyset$ be a closed, covex, bounded subset of a uniformly convex Banach space X, and $T : C \to C$ be asymptotic nonexpansive and such that $\sum(\alpha_n - 1)$ converges. If X has the Opial property (or its norm is Fréchet differentiable) then both Mann and Ishikawa processes converge weakly to a fixed point of T. See slide 19 for def of Opial Property, and Fréchet differentiability.

Asymptotic Pointwise Contractions - APC

A generalisation of Banach Contraction Principle to asymptotic pointwise contractions. Like Banach Principle, this Fixed Point Theorem has three components:

1. Existence of a fixed point
2. Fixed point uniqueness
3. Iterative construction of the fixed point by taking the limit of an orbit

Kirk and Xu (2008)) Fixed Point Theorem

Let $C \neq \emptyset$ be a closed, convex, bounded subset of a reflexive Banach space and $T : C \to C$ be an asymptotic pointwise contraction, i.e. there exists a sequence of mappings $\alpha_n : C \to [0, 1]$ converging pointwise to $\alpha : C \to [0, 1)$ such that $\|T^n(x) - T^n(y)\| \leq \alpha_n(x)\|x - y\|$ for any $x, y \in C$. Then, there exists a unique $z \in C$ such that $T(z) = z$. Moreover, for any $x \in C$, there holds $\|T^n(x) - z\| \to 0$.

Asymptotic Pointwise Nonexpansive Mappings - APNM

This class of mappings is an interesting generalisation of both pointwise contractions and asymptotic nonexpansive mappings. As observed by Kirk, if T^n is continuously Fréchet differentiable on an open convex set containing C for each n, with the Fréchet derivative of T^n at $x \in C$ denoted as $(T^n)'_x$, then T is asymptotic pointwise nonexpansive on C if and only if $\limsup_{n\to\infty} \|(T^n)'_x\| \leq 1$ for each $x \in C$.

Kirk/Xu Fixed Point Theorem (2008)

Let $C \neq \emptyset$ be a closed, covex, bounded subset of a uniformly convex Banach space X, and $T : C \to C$ be asymptotic pointwise nonexpansive, i.e.

$$\|T^n(x) - T^n(y)\| \leq \alpha_n(x)\|x - y\|$$

for all $x, y \in C$, where $\limsup_{n\to\infty} \alpha_n(x) \leq 1$. Then, there exists a $z \in C$ such that $T(z) = z$. The set of all fixed points is closed and convex.

Fixed Point Theorem for APNM idea of proof

The idea is to get a fixed point as a limit of an orbit (like in contraction case) but select a 'smart' starting point. Fix any $x \in C$. Define a type $\varphi(y) = \limsup_{n\to\infty} \|T^n(x) - y\|$. Since types are l.s.c. in Banach spaces and C is weakly compact, then φ attains its inf on a $w \in C$. Use then

The pralellogram inequality for U.C. Banach spaces

$$\left\|\frac{f+g}{2}\right\|^2 \le \frac{1}{2}\|f\|^2 + \frac{1}{2}\|g\|^2 - \frac{1}{4}\lambda(\|f-g\|),$$

where $\lambda : [0,\infty) \to [0,\infty)$ is continuous and $\lambda(t) = 0 \Leftrightarrow t = 0$,

with $f = T^{l+m+n}(x) - T^l(w)$ and $g = f = T^{l+m+n}(x) - T^m(w)$, we get

$$\|T^{l+m+n}(x) - \frac{1}{2}(T^l(w) + T^m(w))\|^2 \le$$

$$\frac{1}{2}(\alpha_l(w)^2\|T^{m+n}(x)-w\|^2+\alpha_m(w)^2\|T^{l+n}(x)-w\|^2)-\frac{1}{4}\lambda(\|T^l(w)-T^m(w)\|)$$

Fixed Point Theorem for APNM idea of proof - continuation

Taking $\limsup$ as $n \to \infty$, and using the fact that φ attains its inf at w we obtain

$$\varphi(w) \leq \varphi\Big(\frac{1}{2}\Big(T^l(w) + T^m(w)\Big)\Big)^2 \leq$$

$$\frac{1}{2}(\alpha_l(w)^2 + \alpha_m(w)^2)\varphi(w) - \frac{1}{4}\lambda(\|T^l(w) - T^m(w)\|),$$

hence, $\lambda(\|T^l(w) - T^m(w)\|) \leq 4\Big(\frac{1}{2}(\alpha_l(w)^2 + \alpha_m(w)^2)\varphi(w) - \varphi(w)\Big) \to 0$ as $l, m \to \infty$, because $a_m(w) \to 1$. By properties of λ, $\{T^l(w)\}$ is norm Cauchy and finally $T^n(w) \to z \in C$. Since $\|T(z) - T^{n+1}(w)\| \leq \alpha_1(z)\|z - T^n(w)\|$, taking $n \to \infty$ we get $\|T(z) - z\| \leq 0$, hence $T(z) = z$.

Open (until recently) problems in the area of the Asymptotic Pointwise Nonexpansive Mappings

1. Are iterative fixed point construction processes convergent for asymptotic pointwise nonexpansive mappings?
2. Can Kirk/Xu fixed point results be generalised to spaces other than Banach spaces?

Kozlowski(2010), Hussain/Khamsi(2009), Khamsi/Kozlowski(2010)

1. Kozlowski proved weak convergence of generalised Mann and Ishikawa processes for asymptotic pointwise nonexpansive mappings in uniformly convex Banach spaces with the Opial property.
2. Hussain/Khamsi generalised Kirk/Xu results to metric spaces with the hyperconvex structure (CAT(0) spaces);
3. Khamsi/Kozlowski generalised Kirk/Xu results to modular function spaces (MFS).

Demiclosedness Principle

Kozlowski Demiclosedness Principle for APNM(2010)

There are many versions of Demiclosedness Principle originating from the 1968 Browder's result. We need the following one: Let $C \neq \emptyset$ be a closed, convex, bounded subset of a uniformly convex Banach space X with Opial Property and $T : C \to C$ be APNM such that $\sum(\alpha_n(x) - 1) < \infty$ and each function $\alpha_n(x)$ is bounded. If $x_n \rightharpoonup w$, and $\|T(x_n) - x_n\| \to 0$, then $T(w) = w$.

Convergence of Mann and Ishikawa processes for APNM

The major difficulty in the APNM case resides in the fact that, unlike for the ANM, T^n are not in general Lipschitzian. Therefore, new notions and techniques are necessary as the ANM methods depend strongly on it. Therefore we introduce the notion of well-defined processes: $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$. Note that because $\lim_{k\to\infty} a_k(x) = 1$ for every $x \in C$, then by a suitable choice of $\{n_k\}$ we can always make an iterative process well defined.

Convergence Results for APNM, Kozlowski (2010)

Let $C \neq \emptyset$ be a closed, convex, bounded subset of a uniformly convex Banach space X with Opial Property and $T : C \to C$ be APNM such that $\sum(\alpha_n(x) - 1) < \infty$ and each function α_n is bounded. Under some reasonable assumptions on the sequences $\{n_k\}$, $\{t_k\}$ and $\{s_k\}$, any well-defined generalized Mann or Ishikawa process converges weakly to a fixed point of T.

Note: This theorem defines constructive algorithms that can be actually implemented.

Idea of proof of Convergence Theorems for APNM

In both cases, the main idea is to prove that the sequence $\{x_k\}$ constructed iteratively is an approximate fixed point sequence, i.e. $\lim_{k\to\infty} \|T(x_k) - x_k\| =$. Consider $y, z \in C$, two weak cluster points of the sequence $\{x_k\}$. There exist then two subsequences $\{y_k\}$ and $\{z_k\}$ of $\{x_k\}$ such that $y_k \rightharpoonup y$ and $z_k \rightharpoonup z$. By the Demiclosedness Principle $T(y) = y$ and $T(z) = z$. By construction of $\{x_k\}$, these limits exist

$$r_1 = \lim_{k\to\infty} \|x_k - y\|,\ r_2 = \lim_{k\to\infty} \|x_k - z\|. \tag{1}$$

Aassume that $y \neq z$. By the Opial property we have

$$\begin{aligned} r_1 = \liminf_{k\to\infty} \|y_k - y\| &< \liminf_{k\to\infty} \|y_k - z\| = r_2 \\ = \liminf_{k\to\infty} \|z_k - z\| &< \liminf_{k\to\infty} \|z_k - y\| = r_1. \end{aligned} \tag{2}$$

The contradiction implies $y = z$, i.e. $\{x_k\}$ has at most 1 weak cluster point. C is weakly compact, which implies $\{x_k\}$ has exactly 1 weak cluster point $w \in C$, i.e. $x_k \rightharpoonup w$. By Demiclosedness Principle again, we get $T(w) = w$.

Strong Convergence Result for APNM

Strong Convergence Result for APNM, Kozlowski (2010)

It is interesting that, provided an asymptotic pointwise nonexpansive mapping is such that T^m is a compact mapping for an $m \geq 1$, both generalized Mann and Ishikawa processes converge strongly to a fixed point of T even without assuming the Opial property.

APPENDIX

Uniform Convexity in Banach spaces

The modulus of convexity in Banach spaces is defines as:

$$\delta(\varepsilon) = inf\left\{1 - \left|\left|\frac{f+g}{2}\right|\right|; \|f\| \leq 1, \|g\| \leq 1, \|f-g\| \geq \varepsilon\right\}$$

A Banach space X is called Uniformly Convex (UC) if $\delta(\varepsilon) > 0$ for exery $\varepsilon > 0$.
Assume that a Banach space X is UC. Then X is reflexive and the following "Paralellogram Inequality"' is true:
For any $d > 0$ there exists a continuous function $\lambda : [0, \infty) \to [0, \infty)$ such that $\lambda(t) = 0 \Leftrightarrow t = 0$, and

$$\left|\left|\frac{x+y}{2}\right|\right|^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}\lambda(\|x-y\|),$$

for all $x, y \in X$ such that $\|x\| \leq d$ and $\|x\| \leq d$.

Opial Property and Fréchet differentiability of norm

Opial Property: A Banach space X is said to have the Opial property if for every sequence $\{x_n\} \subset X$ such that $x_n \rightharpoonup x$ and for any $y \in X$ such that $y \neq x$ there holds $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$. Many Banach spaces have Opial property, there are however spectacular excemptions like L^p for $1 < p \neq 2$ and atomless measure. Fortunately: (1) Opial property can be extended to MFS and cover wide class of function spaces including all L^p, (2) Van Dulst proved in 1982 that any Banach space can be re-normalised to posses the Opial property.

Fréchet differentiability: we say that $F : D \to \mathbb{R}$ is Fréchet differentiable at $x \in D$, where D is open, if the limit

$$\lim_{t\to 0} \frac{F(x+ty) - F(x)}{t}$$

exists and is attained uniformly in $y \in S(X)$.

Modular Function Spaces - MFS

Modular Function, Kozlowski (1988)

MFS - generalization of both function and sequence variants of many spaces, like Lebesgue, Köthe, Orlicz, Musielak-Orlicz, Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii. The importance for applications - the richness of structure of MFS, that - besides being Banach spaces - are equipped with modular equivalents of norm or metric notions. They are also equipped with almost everywhere convergence and convergence in submeasure, and with the natural ordering. In many cases, particularly in applications to integral operators, approximation and fixed point theory, modular type conditions can be more easily verified than their metric or norm counterparts. There are also important results that can be proved only using the MFS apparatus. We will use simplified definitions, for the whole exposition of the MFS framework, see the book by W.M. Kozlowski "'Modular Function Spaces"', M. Dekker 1988.

Regular Convex Function Psudomudulars

Let Σ be a σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, s. th. $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Assume there exists an increasing sequence $K_n \in \mathcal{P}$ s. th. $\Omega = \bigcup K_n$. Denote $\mathcal{E}$ - the space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ the space of all functions $f : \Omega \to [-\infty, \infty]$ s. th. there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \le |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

A convex and even funtion $\rho : \mathcal{M}_\infty \to [0, \infty]$ is called a regular convex function pseudomodular if:

1. $\rho(0) = 0$;
2. ρ is monotone;
3. ρ is orthogonally subadditive;
4. ρ has the Fatou property, i.e. $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
5. ρ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Modular Function Spaces

We say that $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. We define $\mathcal{M} = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty \ \rho - a.e\}$, where each f is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. We say that regular function convex psedomodular ρ is a function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.

MFS - Definition

A modular function space is the vector space L_ρ, defined by $L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}$.

The following formula defines the Luxemurg norm in L_ρ: $\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \leq 1\}$. It can be proved that L_ρ with this norm is a Banach space. Therefore, we can use all Banach space techniques. As said before, however, it is often beneficial to use modular techniques as complementary to or replacement of their norm equivalents.

Some of the MFS notions and techniques

MFS - Some Definitions

(a) We say that $\{f_n\}$ is ρ-convergent to f if $\rho(f_n - f) \to 0$.

(b) A sequence $\{f_n\}$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.

(c) A set $C \subset L_\rho$ is called ρ-closed if for any sequence $\{f_n\}$ in C, the convergence $f_n \to f\ (\rho)$ implies that f belongs to C.

(d) A set $C \subset L_\rho$ is called ρ-bounded if $\sup\{\rho(f - g); f \in C, g \in C\} < \infty$

(e) A set $C \subset L_\rho$ is called ρ-a.e. closed if for any $\{f_n\}$ in C which ρ-a.e. converges to f, then we must have $f \in C$;

(f) A set $C \subset L_\rho$ is called ρ-a.e. compact if for any $\{f_n\}$ in C, there exists a subsequence $\{f_{n_k}\}$ which ρ-a.e. converges to some $f \in C$.

Need to use these notions with caution as modular properties differ sometimes from norm/metric. E.g., the ρ-convergence does not necessarily imply ρ-Cauchy condition, and ρ does not satisfy the triangle inequality. Also, $f_n \to f\ (\rho)$ does not imply in general $\lambda f_n \to \lambda f\ (\rho)$ for $\lambda > 1$.

Uniform continuity of modulars, Property (R)

We say that the function modular ρ is uniformly continuous if for every $\varepsilon > 0$ and $L > 0$ there exists $\delta > 0$ such that $|\rho(g) - \rho(h+g)| \leq \varepsilon$ if $\rho(h) \leq \delta$ and $\rho(g) \leq L$. Function modulars do not have to be uniformly continuous. In many interesting situations this is actually not the case!

Property (R) - plays the central role in the fixed point theory in MFS

We say that L_ρ has property (R) if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection. This simple geometrical property is a modular equivalent of reflexivity!

Property (R) was introduced in 1991 by Khamsi, Kozlowski and Shutao as a replacement of reflexivity to prove fixed point theorems in Orlicz spaces without Δ_2 (note: a reflexive Orlicz space must satisfy Δ_2). They proved that an Orlicz space L^φ has property (R) if and only if φ is very convex.

(Asymptotic) Pointwise Contractions (APC) in MFS

$T : C \to C$ is called a pointwise ρ-contraction if there exists $\alpha : C \to [0, 1)$ such that $\rho(T(x) - T(y)) \le \alpha(x)\rho(x - y)$ for any $x, y \in C, n \ge 1$; and an asymptotic pointwise contraction if there exists a sequence of mappings $\alpha_n : C \to [0, 1]$ such that $\rho(T^n(x) - T^n(y)) \le \alpha_n(x)\rho(x - y)$ for any $x, y \in C$, where $\{\alpha_n\}$ converges pointwise to $\alpha : C \to [0, 1)$.

Fixed Point Theorems (APC) - Khamsi and Kozlowski (2010)

Let the regular convex function modular ρ be uniformly continuous and has property (R) . Let $C \subset L_\rho$ be nonempty, convex, ρ-closed and ρ-bounded. Let $T : C \to C$ be a (asymptotic) pointwise ρ-contraction. Then T has a unique fixed point $x_0 \in C$ and $\{T^n(x)\}$ ρ-converges to x_0 for any $x \in C$.

Idea of proof: Define $\tau(u) = \limsup_{n\to\infty} \rho(T^n(x) - u)$, and - using uniform continuity of ρ - prove that it is ρ-lower semicontinuous in C and hence attains its infimum at some $x_0 \in C$. Using asymptotic pointwise contraction properties, show that $\tau(x_0) = 0$. Hence, $\rho(T^n(x) - x_0) \to 0$. By the ρ-continuity of T, this forces x_0 to be a fixed point of T.

Strong Opial Property in MFS

We will say that L_ρ satisfies the ρ-a.e. Strong Opial property if for every $\{f_n\} \in L_\rho$ which is ρ-a.e. convergent to 0 such that there exists a $\beta > 1$ for which $\sup\{\rho(\beta f_n)\} < \infty$, the following equality holds for any $g \in E_\rho$, $\liminf\limits_{n\to\infty} \rho(f_n + g) = \liminf\limits_{n\to\infty} \rho(f_n) + \rho(g)$. It can be proved that every convex, orthogonally additive function modular has the ρ-a.e. Strong Opial property (e.g. this covers all L^p and Orlicz spaces).

Fixed Point Theorems (APC) - Khamsi and Kozlowski (2010)

Assume that L_ρ has the ρ-a.e. Strong Opial property. Let $C \subset E_\rho$ be a nonempty, ρ-a.e. compact convex subset such that $\sup\{\rho(\beta(x-y)); x, y \in C\} < \infty$, for some $\beta > 1$. Then any $T : C \to C$ (asymptotic) pointwise ρ-contraction has a unique fixed point $x_0 \in C$. Moreover the orbit $\{T^n(x)\}$ converges to x_0, for any $x \in C$.

Note: E_ρ is the $\|\cdot\|_\rho$-closure of $\mathcal{E}$ (the set of simple functions).

Asymptotic Pointwise Nonexpansive Mappings (APNM)

The situation is even more complex if we move to the APNM case, i.e. $T : C \to C$ such that there exists a sequence of mappings $\alpha_n : C \to [0, \infty)$ with $\rho(T^n(f) - T^n(g)) \leq \alpha_n(f)\rho(f - g)$ and $\limsup_{n\to\infty} \alpha_n(f) \leq 1$ for any $f, g \in L_\rho$.

Fixed Point Theorems (APNM) - Khamsi and Kozlowski (2010)

Let ρ be uniformly convex. Let $C \subset L_\rho$ be convex, ρ-closed and ρ-bounded. Then any APNM, $T : C \to C$, has a fixed point.

The main building blocks of our theory are: (a) modular versions of Uniform Convexity (x4), (b) Property (R), (c) The Unique Best Approximant Property: to every $f \in L_\rho$ there exists a unique $g_0 \in C$ such that $\rho(f - g_0) = inf\{\rho(f - g); g \in C\}$; (d) The Parallelogram Property; (e) The Minimizing Sequence Property: any minimizing sequence for a ρ-type defined in a ρ-closed and ρ-bounded set, is ρ-convergent and its limit is independent of the choice of the minimizing sequence. *Note: ρ-types do not have to be ρ-l.s.c*

APNM Fixed Point Theorem in MFS - Idea of proof

The working of our theory can be summarised as follows:

1. The Uniform Convexity Property implies The Unique Best Approximant Property.
2. The Uniform Convexity Property via The Unique Best Approximant Property implies The Property (R).
3. The Uniform Convexity Property implies The Parallelogram Property.
4. The Parallelogram Property implies The Minimizing Sequence Property for type functions when the minimum is strictly positive.
5. The Property (R) implies The Minimizing Sequence Property for type functions when the minimum is equal to zero.
6. The Minimizing Sequence Property for type functions implies the Fixed Point Property for asymptotic pointwise noenxpansive mappings; the modular limit of a minimizing sequence for a type funtion defined by an orbit is an obvious candidate for a fixed point. We proved that this is indeed the case.