Two-Mode Theory of Double Well Interferometry with Bose-Einstein Condensates

Interferometer Process

- The proposed BEC interferometer involves the following process in a regime where a simple two mode theory can be used to interpret the interferometric effects.
- Initially a large number N of bosons are at very low temperature and in the same spin state are trapped in a single potential well in a BEC state, with all the bosons in the lowest mode $\phi_1(r)$ (which is essentially symmetric).
- The trapping potential is changed from a single well into a double well and back again over some suitable time scale.
- The double well potential is in general asymmetric and this leads to interferometric effects, such as in the probability at the end of the interferometric process of bosons being found in the lowest excited mode $\phi_2(r)$ (which is essentially antisymmetric).
- The asymmetry in the trapping potential may be due to gravitational effects for example, and the idea behind the interferometry is to detect such asymmetry effects by measuring the mean number of bosons found in the excited mode or in spatial density patterns.
- BEC interferometry expected to give a $\sqrt{N}$ enhancement in interferometer precision
- Essentially, the interferometric process from $t = 0$ to $t = T$ involves an initial state $|N, 0, 0\rangle$ and a final state $|N - n, n, T\rangle$ representing the transfer of n bosons from the first mode to the second mode.
- The interferometer process is depicted in Figure 1.

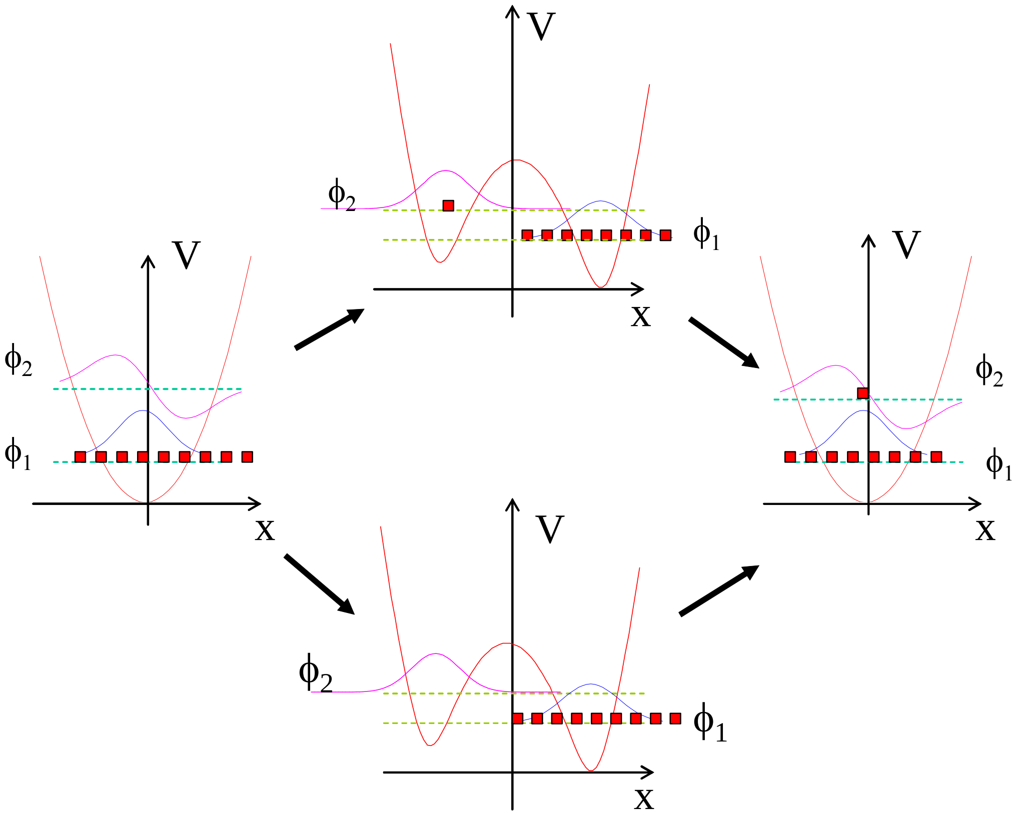

Figure 1. BEC interferometry as a quantum interference process. A trapping potential (shown in red) is changed from a single well into an asymmetric double well and back to a single well again. Initially all the bosons (shown as squares) are in the symmetric lowest mode of the single well, at the end of the process some bosons are in the antisymmetric first excited mode of the single well. Mode functions are depicted in pink and blue, and possible changes to the mode functions during the double well intermediate stage are shown. The case with $N = 9$ bosons initially in mode $\phi_1(r, 0)$ and $n = 1$ bosons finally transferred to mode $\phi_2(r, T)$ is shown. Two quantum pathways are present depending on whether the transfer occurs between $t = 0$ and $t = T/2$ or between $t = T/2$ and $t = T$.

♦ The probability amplitude $A(n, T)$ for the process is related to the transition probability via $P(n, T) = |A(n, T)|^2$ and can be written as the sum of contributions at the intermediate time $T/2$, where m bosons have been transferred from mode $\phi_1(r, 0)$ to mode $\phi_2(r, T/2)$.

♦ Quantum interference in the overall transition amplitude is present, with constructive or destructive interference possible.

Two-Mode Theory

♦ A set of orthogonal, normalized basis states for the N boson system can be defined by

$$\left|\frac{N}{2},k\right\rangle = \frac{(\widehat{c_1}^{\dagger})^{(\frac{N}{2}-k)}}{[(\frac{N}{2}-k)!]^{\frac{1}{2}}}\frac{(\widehat{c_2}^{\dagger})^{(\frac{N}{2}+k)}}{[(\frac{N}{2}+k)!]^{\frac{1}{2}}}|0\rangle \qquad (k=-N/2,-N/2+1,..,+N/2)$$

In general this represents a state with $(\frac{N}{2}-k)$ bosons in mode $\phi_1(r,t)$ and $(\frac{N}{2}+k)$ bosons in mode $\phi_2(r,t)$. Such a state is a fragmented state of the N boson system, involving two BECs not just one. These states are eigenstates with fixed relative boson number $2k$.

♦ The general quantum state $|\Phi(t)\rangle$ of the N boson system during the interferometer process will be written as a quantum superposition of the fragmented states $|\frac{N}{2},k\rangle$, where the amplitude for this fragmented state is $b_k(t)$.

$$|\Phi(t)\rangle = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} b_k(t)\left|\frac{N}{2},k\right\rangle.$$

♦ If the initial condition involves having a single BEC with all bosons in mode $\phi_1(r,0)$ then

$$b_k(0) = \delta_{k,-N/2}$$

♦ The coupled amplitude equations for the $b_k(t)$ are

$$i\hbar\frac{\partial b_k}{\partial t} = \sum_l (H_{kl}-\hbar U_{kl})b_l \qquad (k=-N/2,..,N/2).$$

These $N+1$ equations describe the system dynamics as it evolves amongst the possible fragmented states. The Hamiltonian matrix elements H_{kl} and the rotation matrix elements U_{kl} are defined in terms of the mode functions $\phi_i(r,t)$ and their space and time derivatives.

♦ The coupled Gross-Pitaevskii equations for the two modes $\phi_1(r,t)$ and $\phi_2(r,t)$ are

$$N\sum_j \mu_{ij}\phi_j = \sum_j X_{ij}\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\phi_j + g\sum_{jmn} Y_{imjn}\phi_m^*\phi_n\phi_j \qquad (i = 1,2).$$

The first term represents the kinetic energy of the bosons each of which has mass m, the second term involves the time-dependent trapping potential $V(r,t)$ and the third term allows for the two-body interaction between the bosons in the usual zero-range approximation. The coupling constant g is determined from the scattering length $a_s$via $g = 4\pi a_s\hbar^2/m$. The quantities $\mu_{ij}$are generalised chemical potentials. The quantities X_{ij} and Y_{imjn} are defined by quadratic forms involving the $b_k(t)$.

♦ The amplitudes and modes are determined self-consistently - the mode functions determine the amplitudes and the amplitudes determine the modes.

Fragmentation ?

♦ Whether fragmentation effects occur or not depends on the parameters describing the process.

♦ If no fragmentation occurs the amplitudes are related to binomial coefficients.

Mode Functions - Localised or Delocalised ?

- The form of the mode functions are not pre-determined.
- Possibilities include the mode functions remaining delocalised between the two potential wells or becoming localised in the two different wells.

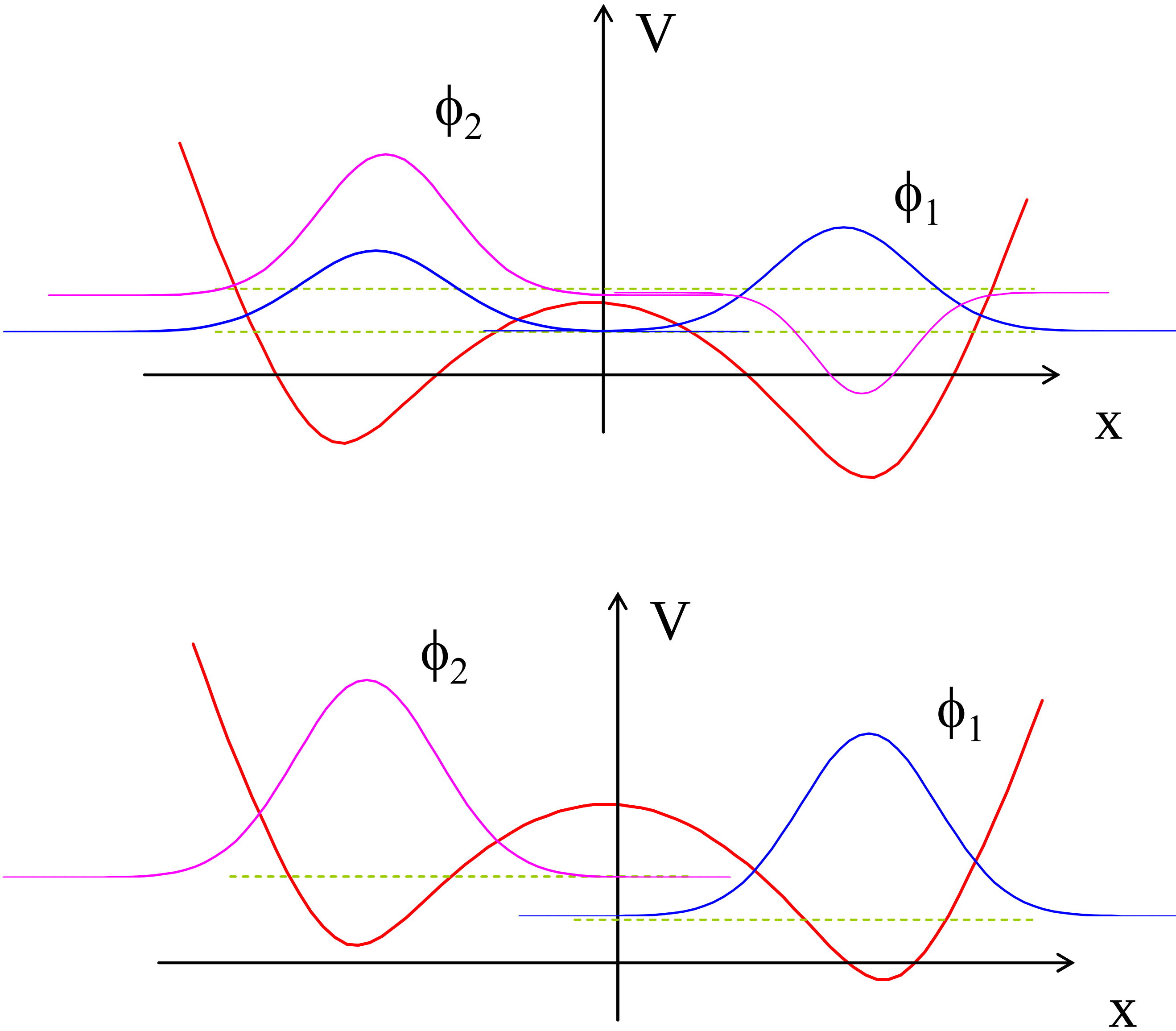

Figure 2. Mode functions in asymmetric trapping potentials showing localization and delocalization effects in the double well regime. For the double well regime with small asymmetry (a) two delocalized modes are shown, one approximately symmetric the other approximately antisymmetric. For the double well regime with large asymmetry (b) two localized modes are shown, each localized in a different well. Boson-boson interactions are ignored.

Interferometric Measurements

♦ The amplitudes $b_k(t)$ and the mode functions $\phi_i(r,t)$ can then be related to the various types of interferometer measurement. For example, the average number of bosons in the mode $\phi_2(r,t)$ is given by

$$\begin{aligned} N_2 &= \left\langle \Phi(t) \right| \hat{c}_2^\dagger(t) \hat{c}_2(t) \left| \Phi(t) \right\rangle \\ &= \frac{\mathrm{N}}{2} + \sum_k k |b_k|^2. \end{aligned}$$

Measurement of N_2 at end of the process depends on the asymmetry and exhibits interferometric effects because the probability amplitude at the end of the process for fragmented states with $k \neq -N/2$ in which there are bosons in the mode $\phi_2(r,t)$ will contain contributions from many quantum pathways.

♦ Interferometric effects of the spatial type can be described in terms of quantum correlation functions. For example, the first order correlation function is given by

$$\begin{aligned} G^{(1)}(\mathbf{r},\mathbf{r}',t) &= \left\langle \Phi(t) \right| \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}(\mathbf{r}') \left| \Phi(t) \right\rangle \\ &= \sum_k b_k{}^* b_k \left\{ \phi_1(\mathbf{r})^* \phi_1(\mathbf{r}') \left(\frac{N}{2} - k \right) + \phi_2(\mathbf{r})^* \phi_2(\mathbf{r}') \left(\frac{N}{2} + k \right) \right\} \\ &\quad + \sum_k b_k{}^* b_{k+1} \left\{ \phi_1(\mathbf{r})^* \phi_2(\mathbf{r}') \sqrt{\left(\frac{N}{2} - k \right)\left(\frac{N}{2} + k + 1 \right)} \right\} \\ &\quad + \sum_k b_k{}^* b_{k-1} \left\{ \phi_2(\mathbf{r})^* \phi_1(\mathbf{r}') \sqrt{\left(\frac{N}{2} + k \right)\left(\frac{N}{2} - k + 1 \right)} \right\}. \end{aligned}$$

The presence of spatial interferometric patterns and the existence of long range order in BECs can be determined from such correlation functions.

Project - PhD

1. Develop computer codes for solving the self-consistent equations for the fragmented state amplitudes and the modes.
2. Solve equations numerically for typical processes
3. Determine the first order quantum correlation function and number of bosons in the excited state at the end of the process.
4. Details are in B J Dalton, Two-mode theory of BEC interferometry, Journal of Modern Optics 54, 615 (2007).

Relative Phase States

♦ Phase angles - Define $N+1$ equispaced relative phase angles θ_p via

$$\theta_p = \frac{2\pi}{\mathrm{N}+1}p \qquad (p = -N/2,\ldots,+N/2)$$

The separation between neighboring angles is small $\frac{2\pi}{\mathrm{N}+1}$ and θ_p essentially ranges from $-\pi$ to $+\pi$.

♦ Orthogonal and normalised phase states are defined by

$$\left|\tfrac{\mathrm{N}}{2},\theta_p\right\rangle = \frac{1}{\sqrt{N+1}}\sum_{k=-\frac{\mathrm{N}}{2}}^{\frac{\mathrm{N}}{2}} \exp(-ik\theta_p)\left|\tfrac{\mathrm{N}}{2},k\right\rangle$$

where $p = -N/2,\ldots,+N/2$, (Pegg & Barnett, PRA 42, 6713 (1990)). The relative phase states and the relative number states are related to each other via Fourier transforms.

♦ Phase operator defined by

$$\widehat{\Theta} = \sum_{p=-\frac{\mathrm{N}}{2}}^{\frac{\mathrm{N}}{2}} \theta_p \left|\tfrac{\mathrm{N}}{2},\theta_p\right\rangle\left\langle\tfrac{\mathrm{N}}{2},\theta_p\right|$$

and hence the phase states $\left|\frac{\mathrm{N}}{2},\theta_p\right\rangle$ are eigenstates of the phase operator with eigenvalues θ_p.

♦ Probability of relative phase measurement determined from

$$A(\theta_p) = \left\langle\tfrac{\mathrm{N}}{2},\theta_p\middle|\Phi(t)\right\rangle$$

$$P(\theta_p) = |A(\theta_p)|^2$$

where $A(\theta_p)$ is the probability amplitude and $P(\theta_p)$ is the probability of measuring relative phase θ_p.

♦ The average relative phase and the variance of the relative phase are defined in terms of the probabilities

$$\overline{\theta_p} = \sum_{p=-\frac{\mathrm{N}}{2}}^{\frac{\mathrm{N}}{2}} \theta_p P(\theta_p)$$

$$(\Delta\theta_p)^2 = \sum_{p=-\frac{\mathrm{N}}{2}}^{\frac{\mathrm{N}}{2}} (\theta_p - \overline{\theta_p})^2 P(\theta_p)$$

♦ The probability amplitudes for relative phase and relative number measurements are related via Fourier transforms

$$A(\theta_p) = \frac{1}{\sqrt{N+1}} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \exp(+ik\theta_p)\, b_k$$

Project - Honours

1. Processes will now be restricted to double well trap potentials, and the mode functions will be assumed fixed as localised functions $\phi_1(r) = \phi_L(r)$ and $\phi_2(r) = \phi_R(r)$ in the left and right wells.
2. The Hamiltonian will now be simplified to that for the Bose-Hubbard model, so that the Hamiltonian matrix elements H_{kl} will now only take into account boson-boson interactions within each well and tunneling between wells. The rotation matrix elements U_{kl} will be zero.
3. Develop computer codes for solving the self-consistent equations for the fragmented state amplitudes.
4. Solve the equations numerically for typical interferometer processes
5. Determine the mean relative phase and its variance.
6. Determine the first order quantum correlation function and relate this to relative phase results.
7. Details are in B J Dalton, Two-mode theory of BEC interferometry, Journal of Modern Optics 54, 615 (2007).